

ON CORNERS SCATTERING STABLY, NEARLY NON-SCATTERING INTERROGATING WAVES, AND STABLE SHAPE DETERMINATION BY A SINGLE FAR-FIELD PATTERN

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ABSTRACT. This article is concerned with two different but correlated topics for time-harmonic acoustic wave scattering from an inhomogeneous medium. The first topic is on corner scattering stably, which sharply quantifies the result in [3] and [22] on corner scattering. In fact, we show that if an inhomogeneous medium scatterer whose support contains a corner is probed by a plane-wave, then the energy of the scattering amplitude possesses a positive lower bound depending only on the size of the corner as well as the refractive index of the medium there. This is evidence against a folk-telling result in the literature: that for a generic inhomogeneous medium, one could generate a set of nearly non-scattering interrogating wave fields by using the Herglotz approximation to the corresponding interior transmission eigenfunctions associated with the inhomogeneous medium. However, the proof of our stability result shows that this folk-telling result might not be true and that it is definitely not so trivial as believed, as long as there is a corner on the support of the inhomogeneous medium scatterer. The second topic is on the stable shape determination of the scattering support of an inhomogeneous medium supported in a polyhedral or polygonal domain, which sharply quantifies the uniqueness result by Hu–Salo–Vesalainen [13]. We establish double-logarithmic stability estimates for the shape determination by a single far-field measurement.

Keywords Helmholtz system, corner scattering, stability, inverse scattering, invisibility

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1. INTRODUCTION

In this paper, we shall be concerned with the direct and inverse problems associated with the time-harmonic acoustic scattering described by the Helmholtz system as follows. Let $k \in \mathbb{R}_+$ be a wavenumber of the acoustic wave, signifying the frequency of the wave propagation. Let $V(x) \in L^\infty(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $n = 2, 3$, be a potential function. $V(x)$ signifies the material parameter of the medium at the point x and it is related to the so-called refractive index in our study. It is assumed that after normalisation, $\text{supp}(V) \subset \Omega$, where Ω is a bounded domain in \mathbb{R}^n . That is, the inhomogeneity of the medium has a compact support.

In the setup of our study, one sends an incident wave field to interrogate the medium V , which is a common means in wave probing. The medium V perturbs the incident wave and create a total wave field. We let u^i and u ,

respectively, denote the incident and total wave fields. u^i is an entire solution to the Helmholtz equation $(\Delta + k^2)u^i = 0$ and u satisfies

$$(\Delta + k^2(1 + V))u = 0, \quad (1.1)$$

in \mathbb{R}^n . Moreover, the scattered wave $u - u^i$ satisfies the Sommerfeld radiation condition

$$|x|^{\frac{n-1}{2}} (\partial_r - ik)(u - u^i) \rightarrow 0, \quad (1.2)$$

uniformly with respect to the angular variable as $r = |x| \rightarrow \infty$, where ∂_r is the derivative along the radial direction from the origin. The radiation condition implies the existence of a far-field pattern. More precisely there is a real-analytic function on the unit-sphere at infinity $A_{u^i} : \mathbb{S}^{n-1} \rightarrow \mathbb{C}$ such that

$$u(r\theta) = u^i(r\theta) + \frac{e^{ikr}}{r^{(n-1)/2}} A_{u^i}(\theta) + \mathcal{O}\left(\frac{1}{r^{n/2}}\right) \quad (1.3)$$

uniformly along the angular variable θ . This function is called the *far-field pattern* or *scattering amplitude* of u . The inverse scattering problem that we are concerned with is to recover V from the knowledge of $A_{u^i}(\theta)$ and the direct scattering problem that we are concerned with is to investigate under what circumstance, one would have $A_{u^i}(\theta) \equiv 0$. This inverse scattering problem serves as a prototype model to many inverse problems arising from scientific and technological applications [6, 17, 31]. The concerned direct scattering problem is related to a significant engineering application, the so-called *invisibility cloaking* (cf. [10, 11, 30]).

Concerning the direct scattering problem described above, it is proved in [3] that if $A_{u^i} \equiv 0$, then the support of V cannot have a 90° corner in \mathbb{R}^n . In [22], it is further shown that under similar conditions, the support of V cannot have a conical corner* in \mathbb{R}^2 or \mathbb{R}^3 . More recently, in [7], the authors show that any penetrable scatterer having a corner or an edge and a real-analytic refractive index function scatters every non-trivial incident wave. The same was shown to be true for less regular functions in [8].

These results mean that a penetrable corner scatters every incident wave non-trivially. Particularly, it implies that one cannot achieve ideal invisibility with a generic regular isotropic medium if there is a corner on the cloaking device. Using the corner scattering result in [3, 7], the authors in [13] considered the inverse scattering problem. It is shown that if two medium scatterers V and V' produce the same scattering amplitude, namely $A_{u^i} = A'_{u^i}$ (where and also in what follows, A'_{u^i} signifies the far-field pattern/scattering amplitude associated with V' corresponding to the incident field u^i), then difference of the supports of V and V' , namely $\text{supp}(V) \triangle \text{supp}(V') := (\text{supp}(V) \setminus \text{supp}(V')) \cup (\text{supp}(V') \setminus \text{supp}(V))$, cannot have a corner of the type that appeared in the previous papers. This means, in particular, that in the set of convex polygonal or rectangular cuboidal penetrable scatterers the far-field pattern produced by sending any single incident wave determines the shape of the scatterer, disregarding its medium content. In what follows, some remarks are in order concerning the significance of the aforementioned results.

*With the exception of a discrete set of opening angles in 3D under which nothing is known so far.

In inverse scattering theory, shape determination by minimal or optimal measurement data remains a longstanding open problem [17]. It has been long conjectured that one can uniquely determine the shape of an impenetrable scatterer by a single far-field measurement. Significant progress has been achieved in recent years in uniquely recovering impenetrable polyhedral scatterers by minimal numbers of far-field measurements; see [1, 5, 20] for related unique recovery results, and [19, 24] for optimal stability estimates. However, the shape determination of a penetrable medium scatterer by a minimal number of far-field measurements is rarely known in the literature. In addition to its significance in inverse scattering theory, the related study is also at the heart of invisibility cloaking, which has received significant attention in the scientific community in recent years due to its practical importance. Blueprints for achieving invisibility via the use of the artificially engineered *metamaterials* were proposed in [12, 18, 23]. Those materials are anisotropic and singular. It is of scientific curiosity and practical importance to know whether one can achieve the invisibility by regular materials. Clearly, the answer is negative if there is a corner on the cloaking device.

In this article, we aim to sharply quantify the aforementioned results in [3, 22] on corner scattering and in [13] on the shape determination by a single far-field pattern. More precisely, we show that for inhomogeneous medium scatterers supported on a polygon, the energy of the scattering amplitude possesses a positive lower bound, depending only on the size of the corner as well as the refractive index of the medium there. As a surprising consequence, we give evidence against a widespread folk-telling result in the literature. Indeed, it is widely believed that one can generate a set of nearly non-scattering interrogating wave fields by approximating the corresponding interior transmission eigenfunction associated with an inhomogeneous medium by Herglotz waves; see, e.g., [4]. As for shape determination, we establish logarithmic estimates in determining the shape of a medium scatter supported in a 2D polygonal or 3D rectangular domain. It can be roughly stated that given two penetrable medium corners V and V' , and a common incident wave u^i , assuming that the far-field patterns of the corresponding scattered waves $u - u^i$ and $u' - u^i$ are ε -close to one another, then the supports of V and V' are $\varphi(\varepsilon)$ -close in the sense of Hausdorff distance. Here, in our study, φ is of doubly logarithmic type.

Due to technical reasons, we first derive the stability estimates in the shape determination by a single far-field pattern. The stability of corner scattering for plane waves follows conveniently then. Our mathematical arguments will be based on quantifying the estimates and coefficients arising in the proofs of [3] and [13]. We state our results in Section 3 and prove them in Section 8. However, there are substantially new challenges and new developments.

The first challenge comes from the different points of view of scatterer probing and always scattering corners: in probing, the philosophy is to use optimal incident waves to recover as much information about the scatterer as possible. In corner scattering, on the other hand, we are given an arbitrary incident wave and want to show non-vanishing of the far-field pattern. We cannot both choose an optimal incident wave and have an arbitrary

one. The associated methods and estimates will work for one, but be overly complicated for the other. Therefor in this first paper on stability we have decided to focus on scatterer probing, which is best done with non-vanishing waves such as plane waves.

A large mathematical challenge is to prove a quantitative Rellich's theorem. As it is well known in scattering theory, a vanishing far-field pattern means a scattered wave that is trivial outside the scattering object [6]. For stability in inverse scattering one requires that a small scattering amplitude corresponds to a small scattered wave. These types of results are well-known for the impenetrable case in the literature [15, 16, 19, 24, 25]. A quantitative Rellich's theorem for the penetrable case is much less studied. Typically, in showing stability for inverse scattering, the far-field data is reduced to the Dirichlet-Neumann map as in [21, 27]. However this map is not well suited for our case since we are interested in *a single incident wave* and the associated far-field pattern of the scattered wave. Therefor we prove a quantitative Rellich's theorem for penetrable scatterers in Section 5. There are two major issues compared to the impenetrable case: a) we do not have a boundary condition for the total wave at the boundary of the scatterer, and b) in the case of support probing the arbitrary positions of the two scatterers causes geometrical issues, for example by producing sharp inwards corners in the union of the two supports. We solve the second issue by propagating the smallness from infinity to the boundary of the convex hull of the two supports and prove geometrical results about the magnitudes of related angles. The earlier issue is more problematic. We cannot use quantitative unique continuation to propagate smallness all the way into the boundary of the convex hull, as the associated function stops being real-analytic there. Dealing with this issue is the source of the two logarithms in our stability estimates.

After solving the two previous challenges the rest of the problems seem like usual smooth sailing: quantifying the norms arising in *complex geometrical optics solutions* estimates that were first considered in [13, 22], estimating the terms arising from the various function expansion first suggested in [3] and combining all into proofs of our theorems.

The structure of the paper is as follows. We define notations in the next section, which will help with stating the main theorems in Section 3. The proof idea is described in Section 4. The quantitative Rellich's theorem and propagation of smallness is proven in Section 5. The fundamental integral identity, along with estimates for its various term is shown in Section 6. The following one, Section 7, has the precise estimates for the complex geometrical optics solutions. Finally after all the ingredients have been prepared, the main theorems are proven in Section 8.

2. NOTATIONS

- (1) We use italic letters P, Q, \dots to denote polytopes, fraktura symbols $\mathfrak{P}, \mathfrak{Q}, \dots$ for polygonal cones, and calligraphic symbols $\mathcal{P}, \mathcal{Q}, \dots$ for spherical cones. This is purely a stylistic choice: all symbols will be defined in the context.

- (2) $B_R = B(\bar{0}, R)$, $0 < R < \infty$: a-priori domain of interest, where the scatterers are located in.
- (3) $P, P' \subset B_R$: the shape of the penetrable scatterers, which are open polytopes.
- (4) $d_H(P, P')$: the Hausdorff distance between the sets P and P' , defined by

$$d_H(P, P') = \max \left(\sup_{x \in P} d(x, P'), \sup_{x' \in P'} d(x', P) \right).$$

- (5) $\|P\|_{T(s,r)}$: a type of norm for the characteristic function χ_P . If it is finite, the latter is a multiplier in the Sobolev space $H_r^s(\mathbb{R}^n)$. See Definition 7.3.
- (6) u^i : incident wave.
- (7) u, u' : corresponding total waves.

Definition 2.1 (Well-posed scattering). A potential $V \in L^\infty(\mathbb{R}^n)$ is said to give a *well-posed scattering problem* if there is a finite \mathcal{S} such that given any incident plane-wave $u^i(x) = \exp(ik\omega \cdot x)$ there is a unique $u \in H_{loc}^2$ such that

$$(\Delta + k^2(1 + V))u = 0$$

and the scattered wave $u^s = u - u^i$ satisfies the Sommerfeld radiation condition. Moreover it has to have the norm bound $\|u^s\|_{H^2(B_{2R})} \leq \mathcal{S}$.

Definition 2.2 (Type 1 admissibility). A potential $V \in L^\infty(B_R)$ is *admissible of type 1* with parameters $\ell, \alpha_m, \alpha_M, s, r, \mathcal{D}, \alpha, \mathcal{M}, \mu, \mathcal{S}$ if the following conditions hold true:

- (1) $V = \chi_P \varphi$ for some open convex polytope $P \subset B_R$ and function φ ,
- (2) the distance from any vertex of P to a non-adjacent edge is at least ℓ ,
- (3) in 2D, P has angles at least $2\alpha_m > 0$ and at most $2\alpha_M < \pi$. In 3D assume that P is a cuboid, i.e. there is a rigid motion taking P to $]0, a[\times]0, b[\times]0, c[$ for some $a, b, c > 0$.
- (4) $\|P\|_{T(s,r)} \leq \mathcal{D}$, and
- (5) $\varphi \in C^\alpha$ with $\|\varphi\|_{C^\alpha} \leq \mathcal{M}$,
- (6) $|\varphi(x_c)| \geq \mu$ for any vertex x_c of P ,
- (7) V gives well-posed scattering with parameter \mathcal{S} .

Definition 2.3 (Type 2 admissibility). We say that a potential $V \in L^\infty(B_R)$ is *admissible of type 2* with parameters $\ell, \alpha_m, \alpha_M, s, r, \mathcal{D}, \alpha, \mathcal{M}, \mu, \mathcal{S}, c, \mathcal{R}$ if, in addition to being admissible of type 1, V satisfies the following conditions:

- (1) if u is the total wave created by any plane-wave $u^i(x) = \exp(ik\omega \cdot x)$ scattering from V , then $|u(x)| \geq c > 0$ for $x \in B_R \setminus P$,
- (2) the total wave u is Lipschitz with positive constant \mathcal{R} in $B_R \setminus P$.

3. STATEMENT OF THE STABILITY RESULTS

Theorem 3.1. *Let $n \in \{2, 3\}$, $k > 0$ and $R > 1$. Let $0 \leq s < \frac{5}{6}$ in 2D or $\frac{1}{4} < s < \frac{3}{4}$ in 3D, and set $r = 2(n+1)/(n+3)$. Let $V, V' \in L^\infty(B_R)$ be admissible of type 2 with parameters $\ell \leq 1, \alpha_m, \alpha_M, s, r, \mathcal{D}, \alpha > s, \mathcal{M}, \mu > 0, \mathcal{S}, c > 0, \mathcal{R}$.*

Let $\mathfrak{h} = d_H(P, P')$ be the Hausdorff distance of P and P' and let $u^i(x) = \exp(ik\omega \cdot x)$ be any plane-wave and $u_\infty^s, u_\infty'^s$ be the far-field patterns of the scattered waves produced by V and V' , respectively.

There are constants $\varepsilon_{\min}, C < \infty$ — which depend on the a-priori parameters only — and $\eta = \eta(\alpha, n, r, s) > 0$ such that if

$$\|u_\infty^s - u_\infty'^s\|_{L^2(\mathbb{S}^{n-1})} < \varepsilon_{\min}$$

then

$$\mathfrak{h} \leq C \left(\ln \ln \frac{\mathcal{S}}{\|u_\infty^s - u_\infty'^s\|_{L^2(\mathbb{S}^{n-1})}} \right)^{-\eta}. \quad (3.1)$$

We remark that in the following theorem the refractive index function φ is allowed to vanish at the vertices. As long as there is one corner where it does not vanish, and the scatterer can fit inside the convex cone generated by that corner, then we can show a lower bound for the scattering amplitude.

Theorem 3.2. *Let $n \in \{2, 3\}$, $k > 0$ and $R > 1$. Let $0 \leq s < \frac{5}{6}$ in 2D or $\frac{1}{4} < s < \frac{3}{4}$ in 3D, and set $r = 2(n+1)/(n+3)$. Let $V \in L^\infty(B_R)$ be admissible of type 1 with parameters $\ell \leq 1$, $\alpha_m, \alpha_M, s, r, \mathcal{D}, \alpha > s, \mathcal{M}, \mu = 0, \mathcal{S}$.*

Recall that any vertex of P is at least distance ℓ from any other vertex or non-adjacent edge of P . Let $u^i(x) = \exp(ik\omega \cdot x)$ be any plane-wave and u_∞^s be the far-field pattern of the scattered wave produced by V .

There are constants $\varepsilon_{\min}, C < \infty$ — which depend on the a-priori parameters except ℓ — and $\gamma = \gamma(\alpha, n, r, s) \geq 6$ such that

$$\|u_\infty^s\|_{L^2(\mathbb{S}^{n-1})} \geq \min \left(\frac{\mathcal{S}}{\exp \exp(C\ell^{-\gamma} |\varphi(x_c)|^{-\gamma-2})}, \varepsilon_{\min} \right). \quad (3.2)$$

4. IDEA OF THE PROOFS

The proof of the stability of scatterer support probing is geometrically much more difficult than showing the lower bound for the scattering amplitude. For this reason we start with support probing. After that it is very convenient to show stability of corner scattering by using the previous lemmas with the second scatterer being identically zero. For support probing we start with the two potentials $V = \chi_P \varphi$, $V' = \chi_{P'} \varphi'$ with P, P' being two open convex polytopes in the domain of interest B_R .

Propagation of smallness plays an important role next. Let $w = u - u'$ be the difference of the total (and hence scattered) waves. Its far-field pattern is the difference of the far-field patterns of u and u' , and hence small when proving stability. We first propagate that smallness into the near-field by an Isakov-type estimate (Proposition 5.2 which is inspired by [15] and [25]). After that we propagate it to near the scatterers V and V' by a chain of balls argument (Proposition 5.7) and then into the scatterers by Hölder continuity (Proposition 5.10).

We will deal with local issues next. We turn our attention to a vertex x_c of let's say P so that $d(x_c, P)$ achieves the Hausdorff distance between P and P' . Consider a relative neighbourhood $P_h = P \cap B(x_c, h)$ of x_c in P for a $h > 0$ small enough. Without going into the details we mention that

bounds for h give bounds for $d_H(P, P')$. We have two representations for the integral

$$\int_{P_h} V(x) u_0(x) u'(x) dx$$

where u_0 is any (possibly nonphysical) solution to

$$(\Delta + k^2(1 + V))u_0 = 0 \quad (4.1)$$

and $u' : \mathbb{R}^n \rightarrow \mathbb{C}$ is the total wave satisfying $(\Delta + k^2(1 + V'))u' = 0$ corresponding to the incident wave u^i . Near P_h it is actually a solution to the constant coefficient equation

$$(\Delta + k^2)u' = 0 \quad (4.2)$$

because $V' = 0$ there.

For the first representation we will use (4.1) and Green's formula. The total wave u satisfies

$$(\Delta + k^2(1 + V))u = 0. \quad (4.3)$$

Integration by parts in a truncated cone Q_h slightly larger than P_h gives

$$\int_{P_h} V(x) u_0(x) u'(x) dx = \int_{\partial Q_h} u_0 \partial_\nu (u - u') - (u - u') \partial_\nu u_0 d\sigma \quad (4.4)$$

by (4.2) and (4.3). For the second representation the a-priori admissibility assumptions and the real-analyticity of u' near P_h imply the splittings

$$V(x) = \varphi(x_c) + \varphi_\alpha(x), \quad |\varphi_\alpha(x)| \leq \|\varphi\|_{C^\alpha(C_h)} |x - x_c|^\alpha,$$

$$u'(x) = u'(x_c) + u'_1(x), \quad |u'_1(x)| \leq \mathcal{R} |x - x_c|.$$

Lastly, we choose $u_0 : \mathbb{R}^n \rightarrow \mathbb{C}$ to be a complex geometrical optics solution

$$u_0(x) = e^{\rho \cdot (x - x_c)} (1 + \psi(x))$$

with $\rho \in \mathbb{C}^n$ such that $\exp(\rho \cdot x)$ decays exponentially in P_h as $|\rho| \rightarrow \infty$. By analysing the complex geometrical optics solution existence proofs of [3], [13] and [22] carefully, and quantifying every step, we see that there are $p \geq 1$ and $\beta > 0$ such that

$$\|\psi\|_{L^p(\mathbb{R}^n)} \leq C |\Im \rho|^{-n/p-\beta} \|V\|$$

where C doesn't depend on ρ or V as long as $|\Im \rho|$ is large enough. The norm $\|V\|$ is a product norm depending on a-priori parameters related to P and φ .

Plug the above function splittings into $\int V u_0 u' dx$ and then estimate all of these integrals in terms of the norms of $u - u'$, $\varphi(x_c)$, $|\Re \rho|$ and h . After that a choice of $|\Re \rho|$ will prove an upper bound for $d_H(P, P')$ based on the smallness ε of the far-field pattern of $u - u'$.

5. FROM THE FAR-FIELD TO THE SCATTERER

The classical Rellich's theorem (Lemma 2.11 in [6]) says that if the far-field pattern of a scattered wave is trivial, then the scattered wave is identically zero outside the support of the potential, perturbation or source term. In this section we will study what is the corresponding result in our case: namely having a penetrable scatterer and a far-field pattern whose norm is small but positive. This kind of question has been studied earlier for impenetrable scatterers by Isakov [15], [16], and more recently by for example Rondi [24] and Liu, Petrini, Rondi, Xiao [19].

Our strategy in this section is as follows. We will first prove a far-field to near-field estimate in the style of Isakov [16] and Rondi, Sini [25]. Then we use an L^∞ three-spheres inequality for solutions to the unperturbed Helmholtz equation $(\Delta + k^2)w = 0$ to propagate smallness from near the boundary of B_{2R} close to the support of the scatterer V . After that use the fact that $w = u - u'$ is Hölder-continuous. This allows the propagation to take the final step, crossing from outside the support of the potentials into the support. Lastly, we use an elliptic regularity estimate to see that the same operations can be done for $w = \nabla(u - u')$.

From the far-field to the near-field. Here we will show that if the far-field patterns A_{u^i} , A'_{u^i} of u and u' are close, then u and u' are close in $B_{2R} \setminus B_R$.

Lemma 5.1. *Let $A, \varepsilon, \mathcal{S} > 0$. Then there is a function $\ell : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for*

$$f(\varepsilon, \ell) = \left(\frac{\ell}{A}\right)^\ell \varepsilon^2 + \mathcal{S}^2$$

we have $f(\varepsilon, \ell(\varepsilon)) \leq 2 \max(\mathcal{S}^2, \varepsilon^2)$. Moreover, when $\varepsilon < \mathcal{S}$ we may set

$$\ell(\varepsilon) = \sqrt{\frac{2}{A} \ln \frac{\mathcal{S}}{\varepsilon}}. \quad (5.1)$$

Proof. If $\varepsilon \geq \mathcal{S}$ choose $\ell(\varepsilon) = A$. Otherwise $\ln(\mathcal{S}/\varepsilon) > 0$ and we may set ℓ as in (5.1), which implies that

$$\frac{\ell}{A} \ln \frac{\ell}{A} \leq \left(\frac{\ell}{A}\right)^2 = \frac{2}{A} \ln \frac{\mathcal{S}}{\varepsilon}$$

i.e. $(\ell/A)^\ell \leq \mathcal{S}^2/\varepsilon^2$ from which the claim follows. \square

The following proposition is inspired by Theorem 4.1 from Rondi and Sini [25], and is more suitable to our case. Their argument is based on the far-field to near-field estimate by Isakov [15] and generalized to any dimension by estimating Hankel functions more carefully than done typically.

Proposition 5.2. *Assume that $w^s \in H_{loc}^2(\mathbb{R}^n)$ satisfies $(\Delta + k^2)w^s = 0$ in $\mathbb{R}^n \setminus \overline{B(0, R)}$ and the Sommerfeld radiation condition at infinity. Let $B_0 > 1$ and $\mathcal{S} \geq 0$ be such that $\|w^s\|_{L^2(B_{2R} \setminus B_R)} \leq \mathcal{S}$*

Let $\varepsilon = \|w_\infty^s\|_{L^2(\mathbb{S}^{n-1})}$, where w_∞^s is the far-field pattern of w^s . Then there is a constant \mathcal{C} depending only on k, R, B_0 such that

$$\|w^s\|_{L^2(B_{2R} \setminus B_R)} \leq \mathcal{C}\varepsilon$$

or

$$\|w^s\|_{L^2(B_{2B_0R} \setminus B_{B_0R})} \leq \mathcal{C} \mathcal{S} B_0^{-\frac{1}{2} \left(\frac{2}{ekR} \ln \frac{\mathcal{S}}{\varepsilon} \right)^{1/2}}.$$

Proof. By the assumptions on w^s it is well known that there is a sequence $b_j > 0$, $j = 0, 1, \dots$ such that its far-field pattern w_∞^s satisfies

$$\|w_\infty^s\|_{L^2(\mathbb{S}^{n-1})}^2 = \sum_{j=0}^{\infty} b_j^2$$

and the function itself has

$$\|w^s\|_{L^2(S(\bar{0}, r))}^2 = \frac{\pi}{2} \sum_{j=0}^{\infty} b_j^2 k r \left| H_{j+(n-2)/2}^{(1)}(kr) \right|^2$$

for any $r > R$. Here $H_\nu^{(1)}$ is a Hankel function of first kind and order ν .

Let $j_0 \in \mathbb{N}$ and $B_0 > 1$. Then

$$\begin{aligned} \|w^s\|_{L^2(S(\bar{0}, r))}^2 &= \frac{\pi}{2} \sum_{j=0}^{j_0} b_j^2 k r \left| H_{j+(n-2)/2}^{(1)}(kr) \right|^2 \\ &\quad + \frac{\pi}{2} \sum_{j=j_0+1}^{\infty} b_j k r \frac{\left| H_{j+(n-2)/2}^{(1)}(kr) \right|^2}{\left| H_{j+(n-2)/2}^{(1)}(kr/B_0) \right|^2} \left| H_{j+(n-2)/2}^{(1)}(kr/B_0) \right|^2 \\ &\leq \frac{\pi}{2} k r \max_{0 \leq j \leq j_0} \left| H_{j+(n-2)/2}^{(1)}(kr) \right|^2 \|w_\infty^s\|_{L^2(\mathbb{S}^{n-1})}^2 \\ &\quad + B_0 \sup_{j > j_0} \frac{\left| H_{j+(n-2)/2}^{(1)}(kr) \right|^2}{\left| H_{j+(n-2)/2}^{(1)}(kr/B_0) \right|^2} \|w^s\|_{L^2(S(\bar{0}, r))}^2 \end{aligned}$$

by the two formulas above. By Corollary 3.8 from Rondi and Sini [25] we see that if $0 < z_1 \leq z_2 < \infty$ then there is $C = C(z_1, z_2) < \infty$ such that

$$\left| H_\nu^{(1)}(z) \right|^2 \leq C^2 \frac{4}{\pi e z} \max \left(1, \left(\frac{2\nu}{ez} \right)^{2\nu-1} \right) \quad (5.2)$$

and

$$C^{-2} \frac{4}{\pi e z} \left(\frac{2\nu}{ez} \right)^{2\nu-1} \leq \left| H_\nu^{(1)}(z) \right|^2 \leq C^2 \frac{4}{\pi e z} \left(\frac{2\nu}{ez} \right)^{2\nu-1} \quad (5.3)$$

for $z_1 \leq z \leq z_2$ and $\nu \in \frac{1}{2}\mathbb{N}$. We will integrate the formula above for $\|w^s\|_{L^2(S(\bar{0}, r))}^2$ along the segment $r \in [B_0R, 2B_0R]$, and so the minimal value of kr/B_0 will be $z_1 := kR > 0$, and the maximal value of the larger kr shall be $z_2 := 2B_0kR < \infty$.

Write $\nu_0 = j_0 + (n-2)/2$ and assume that $\nu_0 \geq 3/2$. Then, using the two Hankel function estimates above, we can continue estimating

$$\|w^s\|_{L^2(S(\bar{0}, r))}^2 \leq \frac{2C^2}{e} \left(\frac{2\nu_0}{ekr} \right)^{2\nu_0-1} \|w_\infty^s\|_{L^2(\mathbb{S}^{n-1})}^2 + C^4 B_0^{1-2\nu_0} \|w^s\|_{L^2(S(\bar{0}, r/B_0))}^2$$

whenever $B_0 R \leq r \leq 2B_0 R$. Next, we integrate $\int_{B_0 R}^{2B_0 R} \dots dr$ to get

$$\begin{aligned} & \|w^s\|_{L^2(B_{2B_0 R} \setminus B_{B_0 R})}^2 \\ & \leq \frac{C^2 R B_0}{e} \frac{1}{\nu_0 - 1} \left(1 - \frac{1}{2^{2\nu_0 - 2}}\right) \left(\frac{2\nu_0}{ekRB_0}\right)^{2\nu_0 - 1} \|w_\infty^s\|_{L^2(\mathbb{S}^{n-1})}^2 \\ & \quad + C^4 B_0^{2-2\nu_0} \|w^s\|_{L^2(B_{2R} \setminus B_R)}^2 \end{aligned}$$

where we have denoted $\bar{0}$ -centered discs of radius ℓ by B_ℓ . Write $\|w_\infty^s\|_{L^2(\mathbb{S}^{n-1})} = \varepsilon$ and recall from the proposition statement that $\mathcal{S} \geq \|w^s\|_{L^2(B_{2R} \setminus B_R)}$. Since $\nu_0 \in \frac{1}{2}\mathbb{N}$, we have $\nu_0 \geq 3/2$, and the inequality above can be continued with

$$\|w^s\|_{L^2(B_{2B_0 R} \setminus B_{B_0 R})}^2 \leq \max\left(\frac{2C^2 R}{e}, C^4\right) \frac{1}{B_0^{2\nu_0 - 2}} \left(\left(\frac{2\nu_0}{ekR}\right)^{2\nu_0 - 1} \varepsilon^2 + \mathcal{S}^2\right).$$

Let $\ell = \sqrt{\frac{2}{ekR} \ln \frac{\mathcal{S}}{\varepsilon}}$. In the remainder of the proof we will consider the case where both $\ell \geq 3$, i.e. $\varepsilon \leq \mathcal{S} \exp(-9ekR/2)$ and $\ell \geq ekR$, i.e. $\varepsilon \leq \mathcal{S} \exp(-e^3 k^3 R^3/2)$. In the other case $\varepsilon > \mathcal{S} \exp(-\max(9ekR, e^3 k^3 R^3)/2)$, and the claim follows directly since the exponential depends only on k, R, B_0 .

We can choose $\nu_0 \in \frac{1}{2}\mathbb{N}$ such that $3/2 \leq \nu_0$ and $2\nu_0 \leq \ell$. Then

$$\left(\frac{2\nu_0}{ekR}\right)^{2\nu_0 - 1} \leq \left(\frac{\ell}{ekR}\right)^{2\nu_0 - 1} \leq \left(\frac{\ell}{ekR}\right)^\ell$$

since $\ell \geq 2\nu_0$, $2\nu_0 - 1 \geq 0$ and $\ell/(ekR) \geq 1$. The inequality $\varepsilon < \mathcal{S}$ is true by the assumptions from the previous paragraph. Lemma 5.1 implies then that

$$\|w^s\|_{L^2(B_{2B_0 R} \setminus B_{B_0 R})}^2 \leq \max\left(\frac{2C^2 R}{e}, C^4\right) \frac{2\mathcal{S}^2}{B_0^{2\nu_0 - 2}}.$$

We make our final choice, and set $\nu_0 = \lfloor \ell \rfloor / 2$. Then $2\nu_0 > \ell - 1$ which implies

$$\|w^s\|_{L^2(B_{2B_0 R} \setminus B_{B_0 R})}^2 \leq \max\left(\frac{2C^2 R}{e}, C^4\right) \frac{2\mathcal{S}^2 B_0^3}{B_0^\ell}$$

i.e. the claim follows. \square

Corollary 5.3. *Let $w^s \in H_{loc}^2(\mathbb{R}^n)$ satisfy $(\Delta + k^2)w^s = 0$ in $\mathbb{R}^n \setminus \overline{B}(\bar{0}, R)$ and the Sommerfeld radiation condition at infinity. Let w_∞^s be its far-field pattern.*

Assume that there is $\mathcal{S} \geq 0$ such that $\|w^s\|_{L^2(B_{2R} \setminus B_R)} \leq \mathcal{S}$ and denote $\varepsilon = \|w_\infty^s\|_{L^2(\mathbb{S}^{n-1})}$. Let A be a domain such that $\overline{A} \subset B_{2R} \setminus \overline{B}_R$. Then, for $r \in \mathbb{N}$, there are constants $c, C > 0$ depending only on k, r, R, A such that

$$\|w^s\|_{H^r(A)} \leq C \max\left(\varepsilon, \mathcal{S} e^{-c\sqrt{\ln(\mathcal{S}/\varepsilon)}}\right).$$

Proof. We will use elliptic interior regularity to prove the claim. Firstly, if $f \in \mathcal{S}'(\mathbb{R}^n)$ then

$$\|f\|_{H^{s+2}(\mathbb{R}^n)} = \|(1 + k^2)f - (\Delta + k^2)f\|_{H^s(\mathbb{R}^n)}$$

for any $s \in \mathbb{R}$. Let $B_0 > 1$ be such that $\overline{A} \subset \Omega := B_{2R} \setminus \overline{B}_{B_0 R}$. If $\varphi \in C_0^\infty(\Omega)$ we have

$$\|(\Delta + k^2)(\varphi w^s)\|_{H^s(\mathbb{R}^n)} = \|2\nabla\varphi \cdot \nabla w^s + w^s \Delta\varphi\|_{H^s(\mathbb{R}^n)} \leq C_\varphi \|w^s\|_{H^{s+1}(\Omega)}.$$

Here w^s was extended by zero outside of Ω . Let $\Omega' \subset \Omega$ be a subdomain a positive distance from the boundary of Ω . Now, if we have $\varphi \equiv 1$ on Ω' , then

$$\begin{aligned} \|w^s\|_{H^{s+2}(\Omega')} &\leq \|\varphi w^s\|_{H^{s+2}(\mathbb{R}^n)} \leq (1+k^2)C_\varphi \|w^s\|_{H^s(\Omega)} + C_\varphi \|w^s\|_{H^{s+1}(\Omega)} \\ &\leq C_{k,\varphi} \|w^s\|_{H^{s+1}(\Omega)}. \end{aligned}$$

by the two equations above.

Next, the proposition implies

$$\|w^s\|_{L^2(\Omega)} \leq \mathcal{C} \max \left(\varepsilon, \mathcal{S} B_0^{-\frac{1}{2} \left(\frac{2}{ekR} \ln \frac{\mathcal{S}}{\varepsilon} \right)^{1/2}} \right)$$

directly. Given $r \in \mathbb{N}$ take a sequence $A = \Omega_r \subset \Omega_{r-1} \subset \dots \subset \Omega_0 = \Omega$ of sets whose boundaries are a positive distance apart. Also, take a sequence of smooth cutoff functions $\varphi_j \in C_0^\infty(\Omega_j)$ such that $\varphi_j \equiv 1$ on Ω_{j+1} . Then we use the last estimate of the previous paragraph inductively to get

$$\|w^s\|_{H^r(\Omega_r)} \leq C_{k,\varphi_0,\dots,\varphi_{r-1}} \mathcal{C} \max \left(\varepsilon, \mathcal{S} B_0^{-\frac{1}{2} \left(\frac{2}{ekR} \ln \frac{\mathcal{S}}{\varepsilon} \right)^{1/2}} \right)$$

from the $L^2(\Omega)$ -norm of w^s . \square

Three spheres inequality and chain of balls. We state an L^∞ three-balls inequality for solutions to the Helmholtz equation. It follows from Lemma 3.5 in [24] by suitable choices of parameters. After that we prove a few lemmas and a proposition which will allow us to propagate the smallness from outside a large ball along a straight line to near the scatterers V and V' .

Lemma 5.4. *There are positive constants R_m, C, c_1 such that $0 < c_1 < 1$, which depend only on k and satisfy the following: Let $x \in \mathbb{R}^n$ and $0 < 4r < R_m$. If w satisfies*

$$(\Delta + k^2)w = 0$$

in $B_{4r} := B(x, 4r)$, then

$$\|w\|_{B_{2r}} \leq C(2 + \sqrt{2})^{\frac{3}{2}} \|w\|_{B_{4r}}^{1-\beta} \|w\|_{B_r}^\beta \quad (5.4)$$

where the norms are L^∞ -norms in the corresponding x -centered balls and β is a number that satisfies

$$\frac{c_1}{4} \leq \beta \leq 1 - \frac{3c_1}{4}.$$

Proof. Choose $\rho_1 = r$, $\rho = 2r$, $\rho_2 = 4r$, $\tilde{\rho}_0 = R_m$ and $s = 2^{3/2}r$ in Lemma 3.5 of [24]. Also choose $u(\cdot) = w(\cdot - x)$. \square

Lemma 5.5. *Let $K \in \mathbb{N}$, $r > 0$ and B_1, \dots, B_K be a chain of balls with the following properties:*

- (1) $4r < R_m$, defined in Lemma 5.4,
- (2) the radii of each B_k is r ,
- (3) the center-center distance from B_k to B_{k+1} is $\leq r$.

Let $U \subset \mathbb{R}^n$ be open and $w \in L^\infty(U)$ satisfy the Helmholtz equation $(\Delta + k^2)w = 0$ there, and $\|w\|_{L^\infty(U)} \leq \mathcal{T}$ which we assume to be at least 1. Assume that each $B_k \subset U$ and moreover that $d(B_k, \partial U) \geq 3r$.

Then there are finite $C \geq 1$, $0 < c_2 < 1/4$ depending only on k such that

$$\|w\|_{B_K} \leq C\mathcal{T} \|w\|_{B_1}^{c_2^{K-1}}$$

if $\|w\|_{B_1} \leq 1$, where the norms are the L^∞ -norms in the corresponding balls.

Proof. Lemma 5.4 and the fact that B_{K-k} is covered by the $2r$ -radius ball with same center as B_{K-k-1} implies that

$$\|w\|_{B_k} \leq C(2 + \sqrt{2})^{3/2} \mathcal{T}^{1-\beta} \|w\|_{B_{k-1}}^\beta.$$

Estimate $\|w\|_{B_K}$ as above and continue telescopically to get

$$\|w\|_{B_K} \leq C^{1+\beta+\dots+\beta^{K-2}} (2+\sqrt{2})^{\frac{3}{2}(1+\beta+\dots+\beta^{K-2})} \mathcal{T}^{(1-\beta)(1+\beta+\dots+\beta^{K-2})} \|w\|_{B_1}^{\beta^{K-1}}.$$

Note that $1 + \dots + \beta^{K-2} \leq 1/(1-\beta) \leq 4/(3c_1)$ and $\beta \geq c_1/4$. The claim follows by setting $c_2 = c_1/4$. \square

Corollary 5.6. *Let $U \subset \mathbb{R}^n$ be open, $w \in L^\infty(U)$ such that $(\Delta + k^2)w = 0$. Let $\gamma \subset U$ be a rectifiable curve between two different points $x, x' \in U$ such that $B(\gamma, 4r) \subset U$ for some $r > 0$. Assume that the L^∞ -norms satisfy $\|w\|_{B(x,r)} \leq 1$ and that $\|w\|_U \leq \mathcal{T}$ which is at least one.*

Then for any $y \in \gamma$ we have

$$\|w\|_{B(y,r)} \leq C\mathcal{T} \|w\|_{B(x,r)}^{d_\gamma(x,y)/r+1} \leq C\mathcal{T} \|w\|_{B(x,r)}^{d_\gamma(x,x')/r+1}$$

if $4r \leq R_m$ as in Lemma 5.4. Here d_γ is the distance measured along γ .

Proof. Denote $l = d_\gamma(x, y)$. We build a sequence of balls with radii r and centers $x_1 = x, x_2, x_3, \dots, x_{\lceil l/r \rceil}$, and set $x_{\lceil l/r \rceil+1} = y$. For all these points except the first and the last x_{k+1} is distance r from x_k along γ , and hence distance at most r in \mathbb{R}^n . For example if $l = 2r$ we would get the triple x, x_2, y with $2 = \lceil l/r \rceil$. For $l = (2 + \frac{1}{2})r$ we would get the 4-tuple x, x_2, x_3, y with $3 = \lceil l/r \rceil$. Then use the previous lemma with $B_k = B(x_k, r)$ and $K = \lceil l/r \rceil + 1 \leq l/r + 2$. Since $\|w\|_{B(x,r)} \leq 1$ and $c_1/4 < 1$ both estimates follow. \square

We are now ready to state and prove the propagation of smallness in the context of corner scattering. Recall that P and P' contain the supports of the potentials V, V' , and both are contained in $B_R = B(\bar{0}, R)$ for some fixed $R > 0$. Moreover both are convex. This is important to ensure that $B_R \setminus (P \cup P')$ is simply connected.

Proposition 5.7. *Let $Q \subset B_R \subset \mathbb{R}^n$ be a convex polygon. Let w be a function such that $w \in L^\infty(B_{2R} \setminus Q)$ satisfies $(\Delta + k^2)w = 0$ in that same set, with L^∞ -norm at most $\mathcal{T} \geq 1$. Let $4r \leq R_m$ from Lemma 5.4 and $2r < (1 - 2\lambda)R$ for some positive $\lambda < \frac{1}{2}$.*

Assume that $\|w\|_{L^\infty} \leq \delta \leq 1$ in $B_{(2-\lambda)R} \setminus B_{(1+\lambda)R}$. Then

$$\|w\|_{L^\infty(B_{2R} \setminus B(Q, 4r))} \leq C\mathcal{T} \delta^{c_2^{(2+\lambda)R/r+2}}$$

where $C \geq 1$ and $0 < c_2 < 1/4$ are as in Lemma 5.5.

Proof. Let $x' \in B_{2R} \setminus B(Q, 4r)$. Since Q is convex there is a ray from x' into $B_{2R} \setminus B_{(1+\lambda)R}$ that's at least distance $4r$ from Q . It can be constructed as follows: consider the line from $\bar{0}$ to x' (if $x' = \bar{0}$ any line is fine). The point x' splits it into two rays. Take the one not touching $B(Q, 4r)$.

Cut a segment from the ray, starting at x' and ending distance r outside $B_{(1+\lambda)R}$ to make sure that $\|w\|_\infty \leq \delta$ in the first ball in the chain of balls we are about to use. This ball has radius r and since $2r < (1 - 2\lambda)R$ it fits completely inside $B_{(2-\lambda)R} \setminus B_{(1+\lambda)R}$. The length of that segment is then at most $R + (1 + \lambda)R + r$. Then use Corollary 5.6. \square

Propagation of smallness into the perturbation. The following proposition will be used to estimate $u - u'$ and $\nabla u - \nabla u'$ in Lemma 6.2. To be able to do that, we will later show that both of these differences are Hölder-continuous. The case of $u - u'$ follows directly from Sobolev embedding in \mathbb{R}^2 and \mathbb{R}^3 because $V, V' \in H^s(\mathbb{R}^n)$ for $s < 1/2$. The smoothness of the gradient will follow from elliptic regularity estimates for boundary value problems with smooth boundary values. After all, $u - u'$ is real analytic outside of the supports of the potentials V and V' .

Proposition 5.8. *Let $Q \subset B_R \subset \mathbb{R}^n$ be a convex polygon. Let $w \in L^\infty(B_{2R})$ be such that $w \in C^\alpha(\bar{B}_{3R/2})$ with norm at most $\mathcal{T} \geq 1$ for some $0 < \alpha < 1$ and it satisfies $(\Delta + k^2)w = 0$ in $B_{2R} \setminus Q$.*

Assume that $|w(x)| \leq \delta$ in $B_{(2-\lambda)R} \setminus B_{(1+\lambda)R}$ for some positive $\lambda < \frac{1}{2}$ and let $A \geq 2 + \lambda$. If

$$\delta < 1 / \exp \exp \left(\frac{4AR |\ln c_2| / (1 - \alpha)}{\max(R_m, R/2, 2(1 - 2\lambda)R)} \right) \quad (5.5)$$

where R_m is given in Lemma 5.4 then

$$|w(x)| \leq \frac{(8AR |\ln c_2| / (1 - \alpha))^\alpha + C/c_2^2 \mathcal{T}}{(\ln |\ln \delta|)^\alpha} \quad (5.6)$$

for $x \in B_R$ satisfying $d(x, \partial Q) \leq 4AR |\ln c_2| / ((1 - \alpha) \ln |\ln \delta|)$. Here C and c_2 are given by Lemma 5.5.

Proof. Choose

$$r = r(\delta) = \frac{AR |\ln c_2|}{(1 - \alpha) \ln |\ln \delta|}$$

with c_2 from Lemma 5.5. Then $0 < r$. By the upper bound on δ we have $4r < R_m$ and $2r < (1 - 2\lambda)R$ as required in Proposition 5.7. By that same proposition

$$|w(x)| \leq C\mathcal{T}\delta^{c_2^{(2+\lambda)R/r+2}}$$

when $x \in B_{2R}$, $d(x, Q) \geq 4r$.

Let $d(x', \partial Q) \leq 4r$ now. By convexity any boundary point of Q is distance $4r$ from $\mathbb{R}^n \setminus B(Q, 4r)$. The upper bound on δ implies $R + 4r < 3R/2$, and so $\bar{B}(Q, 4r) \subset B_{3R/2}$. Hence there is $x \in B_{3R/2} \setminus B(Q, 4r)$ such that $|x - x'| \leq 8r$. By the Hölder continuity of w we have

$$|w(x')| \leq \|w\|_{C^\alpha(\bar{B}_{3R/2})} |x - x'|^\alpha + |w(x)| \leq \mathcal{T} 8^\alpha r^\alpha + C\mathcal{T}\delta^{c_2^{2R/r+2}}.$$

The choice of $r = r(\delta)$ implies that

$$r^\alpha = \frac{(AR |\ln c_2| / (1 - \alpha))^\alpha}{(\ln |\ln \delta|)^\alpha},$$

and

$$\delta^{c_2^{(2+\lambda)R/r+2}} = e^{-|\ln \delta| c_2^{(2+\lambda)R/r+2}} = e^{-c_2^2 |\ln \delta|^{1-(2+\lambda)(1-\alpha)/A}}.$$

Now, since $A \geq 2 + \lambda$, we can continue with

$$\dots \leq e^{-c_2^2 |\ln \delta|^\alpha} \leq \frac{1}{c_2^2 |\ln \delta|^\alpha} \leq \frac{1}{c_2^2 (\ln |\ln \delta|)^\alpha}$$

since $|\ln \delta| > 1$. The claim follows. \square

Quantitative Rellich's theorem.

Lemma 5.9. *Let $n \in \{2, 3\}$ and $q \in L^\infty(B_{2R})$ be supported in B_R for some $R > 0$. Assume that $\|q\|_{L^\infty} \leq \mathcal{M} < \infty$. Let $w \in H^2(B_{2R})$ have H^2 -norm at most $\mathcal{S} < \infty$, and assume that*

$$(\Delta + k^2(1 + q))w = 0.$$

Then $w \in C^{1, \frac{1}{2}}(\overline{B}_{3R/2})$ and there is $C = C(R, k, n)$ such that

$$\|w\|_{C^{1, \frac{1}{2}}} \leq C(1 + \mathcal{M})\mathcal{S}.$$

Proof. Interior elliptic regularity in the domain where $q \equiv 0$ (e.g. Theorem 8.10 by Gilbarg and Trudinger [9]) implies that $w \in H^s(B_{7R/4} \setminus B_{5R/4})$ with $s = (n + 3)/2$ and a corresponding norm estimate. Adding Sobolev embedding gives then

$$\|w\|_{C^{1, \frac{1}{2}}(B_{7R/4} \setminus B_{5R/4})} \leq C \|w\|_{H^{\frac{n+3}{2}}(B_{7R/4} \setminus B_{5R/4})} \leq C \|w\|_{H^2(B_{2R} \setminus B_R)} \quad (5.7)$$

for some other constant $C = C(R, n, k)$. This implies that w has boundary values in $C^{1, \frac{1}{2}}(\partial B_{3R/2})$, i.e. more precisely that there is $\varphi \in C^{1, \frac{1}{2}}(\mathbb{R}^n)$ supported in $B_{7R/4} \setminus B_{5R/4}$ such that $w = \varphi$ on $\partial B_{3R/2}$.

Theorem 8.34 in [9] gives unique solvability for the Dirichlet problem $\Delta v = -k^2(1 + q)w$, $v|_{\partial B_{3R/2}} = \varphi|_{\partial B_{3R/2}}$, in the space of $C^{1, \frac{1}{2}}$ functions with $C^{1, \frac{1}{2}}$ boundary values. This, along with the H^1 maximum principle for the Laplace equation shows that w restricted to $\overline{B}_{3R/2}$ is $C^{1, \frac{1}{2}}$ -smooth. Moreover Theorem 8.33 in [9] gives an estimate for $\|v\|_{C^{1, \frac{1}{2}}(\overline{B}_{3R/2})}$ based on the boundary and source data. Using that, the Sobolev embedding $H^2 \hookrightarrow L^\infty$ and (5.7) gives

$$\|w\|_{C^{1, \frac{1}{2}}(\overline{B}_{3R/2})} \leq C \left(1 + \|1 + q\|_{L^\infty(B_{2R})}\right) \|w\|_{H^2(B_{2R})}$$

for some constant $C = C(R, n, k)$ and the claim follows. \square

Proposition 5.10. *Let $R > 1$, $n \in \{2, 3\}$ and $k > 0$. Let $u^i \in H_{loc}^2(\mathbb{R}^n)$ be an incident wave, $(\Delta + k^2)u^i = 0$.*

Let $P, P' \subset B_R$ be open convex polyhedra, and $\varphi, \varphi' \in L^\infty(\mathbb{R}^n)$. Let $V = \chi_P \varphi$ and $V' = \chi_{P'} \varphi'$ be two potentials with $\|V\|_\infty, \|V'\|_\infty \leq \mathcal{M}$. Also, let $u, u' \in H_{loc}^2(\mathbb{R}^n)$ be total waves satisfying

$$(\Delta + k^2(1 + V))u = (\Delta + k^2(1 + V'))u' = 0$$

and whose scattered waves $u^s = u - u^i$, $u'^s = u' - u^i$ satisfy the Sommerfeld radiation condition. Let $u_\infty^s, u_\infty'^s : \mathbb{S}^{n-1} \rightarrow \mathbb{C}$ be their far-field patterns.

Assume that $\|u - u'\|_{H^2(B_{2R})} \leq \mathcal{S}$ and $\mathcal{S} \geq 1$. There is $\varepsilon_m = \varepsilon_m(\mathcal{S}, k, R) > 0$ such that if

$$\|u_\infty^s - u_\infty'^s\|_{L^2(\mathbb{S}^{n-1})} \leq \varepsilon_m$$

and Q is the convex hull of P and P' then $u - u', \nabla u - \nabla u'$ are continuous in B_R and

$$\sup_{\partial Q} |u - u'| + |\nabla u - \nabla u'| \leq C \left(\ln \ln(\mathcal{S} \|u_\infty^s - u_\infty'^s\|_{L^2(\mathbb{S}^{n-1})}^{-1}) \right)^{-1/2}$$

for some $C = C(\mathcal{M}, \mathcal{S}, k, R)$.

Proof. Firstly, propagate smallness from the far-field to the near-field by using Corollary 5.3. Let w^s in that proposition be $u - u' = u^s - u'^s$ and denote $\varepsilon = \|u_\infty^s - u_\infty'^s\|_{L^2(\mathbb{S}^{n-1})}$. Choose the annulus $A = B_{(2-\lambda)R} \setminus \overline{B_{(1+\lambda)R}}$ for some positive $\lambda < \frac{1}{2}$. By the Sobolev embedding

$$\|w^s\|_{L^\infty(A)}, \|\nabla w^s\|_{L^\infty(A)} \leq C \max \left(\varepsilon, \mathcal{S} e^{-c\sqrt{\ln(\mathcal{S}/\varepsilon)}} \right)$$

for $C > 1, c > 0$ depending on k, R, λ since the number of derivatives required is fixed by the Sobolev embedding and the fact that $n \in \{2, 3\}$. Our first requirement on ε_m is that the maximum picks the number on the right side. This happens if $\varepsilon_m \leq \mathcal{S} e^{-c^2}$.

The second step is to use the propagation of smallness Proposition 5.8 for $w = w^s$ and $w = \partial_j w^s$, $j = 1, \dots, n$. By Lemma 5.9 we have $\|w\|_{C^{1/2}} \leq 2C(1 + \mathcal{M})\mathcal{S}$ in $\overline{B_{3R/2}}$ for each choice of w . Also $C = C(k, R)$. Set

$$\delta = C\mathcal{S} e^{-c\sqrt{\ln(\mathcal{S}/\varepsilon)}}.$$

We get a second upper bound on ε_m by requiring that δ satisfies (5.5). The right-hand side in that inequality depends only on $A = A(\lambda, R)$, k and R , so ε_m still only depends on λ, k, R . Now Proposition 5.8 implies

$$|w(x)| \leq C(1 + \mathcal{M})\mathcal{S}(\ln |\ln \delta|)^{-1/2}$$

with $C = C(\lambda, k, R)$. The choice of δ implies

$$|\ln \delta| = c\sqrt{\ln(\mathcal{S}\varepsilon^{-1})} - \ln(C\mathcal{S}) \geq \frac{c}{2}\sqrt{\ln(\mathcal{S}\varepsilon^{-1})} \geq (\ln(\mathcal{S}\varepsilon^{-1}))^{1/4}$$

if ε_m is small enough (and again c, C depend on k, λ, R). Thus

$$(\ln |\ln \delta|)^{-1/2} \leq \left(\ln (\ln(\mathcal{S}\varepsilon^{-1}))^{1/4} \right)^{-1/2} = 2(\ln \ln(\mathcal{S}\varepsilon^{-1}))^{-1/2}$$

and the claim follows after choosing λ as a function of R for example. \square

6. FROM BOUNDARY TO INSIDE

We deal with particulars related to corner scattering in this section. More precisely, we prove the fundamental orthogonality identity which is the foundation upon which past results [3, 7, 8, 13, 22] were built on. Since we are proving stability instead of uniqueness we will have an extra boundary term here to deal with. Moreover, for future convenience, we will not assume that $u^i(x_c) \neq 0$ in Corollary 6.2. This will not complicate the argument by much.

Proposition 6.1. *Let $Q_h \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $V \in L^\infty(Q_h)$, $k > 0$ and $u^i, u, u_0 \in H^2(Q_h)$ satisfy*

$$\begin{aligned} (\Delta + k^2)u^i &= 0, \\ (\Delta + k^2(1 + V))u &= 0, \\ (\Delta + k^2(1 + V))u_0 &= 0 \end{aligned}$$

in Q_h . Then

$$k^2 \int_{Q_h} V u_0 u^i dx = \int_{\partial Q_h} (u_0 \partial_\nu (u^i - u) - (u^i - u) \partial_\nu u_0) d\sigma. \quad (6.1)$$

Proof. Use Green's formula after noting that

$$k^2 \int_{Q_h} V u_0 u^i dx = \int_{Q_h} u_0 (\Delta + k^2(1 + V))(u^i - u) dx.$$

□

We consider only incident waves that do not vanish anywhere in this paper. This means that in the following corollary we would always have P_N a constant and $N = 0$. The corollary is stated so that it applies also to the more general case where the incident wave can vanish up to a finite order N at x_c . This is for the convenience of future papers on the topic and also since the proof is not substantially more difficult in this case.

Corollary 6.2. *Let $\mathfrak{P}, \mathfrak{Q} \subset \mathbb{R}^n$ be open cones with vertex x_c such that $\mathfrak{P} \subset \mathfrak{Q}$ and their boundaries are a subset of the union of at most \mathcal{V} hyperplanes of codimension 1. Let $P_h = \mathfrak{P} \cap B(x_c, h)$ and $Q_h = \mathfrak{Q} \cap B(x_c, h)$ for $0 < h \leq 1$.*

Let $k > 0$ and $V \in L^\infty(\mathbb{R}^n)$ be such that $V = \chi_{\mathfrak{P}} \varphi$ in Q_h for some measurable function $\varphi : P_h \rightarrow \mathbb{C}$. Let $u^i, u, u_0 \in H^2(Q_h)$ satisfy the partial differential equations in the statement of Proposition 6.1. If we have functions $P_N, \varphi_\alpha, u_{N+1}^i, \psi$ and a complex vector $\rho \in \mathbb{C}^n$ such that

$$\begin{aligned} \varphi(x) &= \varphi(x_c) + \varphi_\alpha(x), \\ u^i(x) &= P_N(x - x_c) + u_{N+1}^i(x), \\ u_0(x) &= e^{\rho \cdot (x - x_c)} (1 + \psi(x)), \end{aligned}$$

in P_h , then

$$\begin{aligned} \varphi(x_c) \int_{\mathfrak{P}} e^{\rho \cdot (x - x_c)} P_N(x - x_c) dx &= \varphi(x_c) \int_{\mathfrak{P} \setminus P_h} e^{\rho \cdot (x - x_c)} P_N(x - x_c) dx \\ &\quad - \int_{P_h} e^{\rho \cdot (x - x_c)} \varphi_\alpha(x) P_N(x - x_c) dx - \int_{P_h} e^{\rho \cdot (x - x_c)} V(x) u_{N+1}^i(x) dx \\ &\quad - \int_{P_h} e^{\rho \cdot (x - x_c)} V(x) u^i(x) \psi(x) dx + \frac{1}{k^2} \int_{\partial Q_h} u_0 \partial_\nu (u - u^i) - (u - u^i) \partial_\nu u_0 d\sigma. \end{aligned} \quad (6.2)$$

If we assume moreover that $\psi \in L^p(Q_h)$, that there is $R > 0$ such that $Q_h \subset B_R$ and $\psi \in H^2(B_{2R})$ and that

- (1) $|\rho| \geq 1$ and $\Re \rho \cdot (x - x_c) \leq -\delta_0 |x - x_c| |\Re \rho|$ for some $\delta_0 > 0$ and any $x \in Q_h$,
- (2) $|\varphi_\alpha(x)| \leq \mathcal{M} |x - x_c|^\alpha$, $|V(x)| \leq \mathcal{M}$ for $x \in P_h$, and some $\alpha > 0$

- (3) $|u^i(x)| \leq \mathcal{F} |x - x_c|^N$ for $x \in P_h$,
- (4) $|P_N(x - x_c)| \leq \mathcal{P} |x - x_c|^N$ for $x \in P_h$,
- (5) $|u_{N+1}^i(x)| \leq \mathcal{R} |x - x_c|^{N+1}$ for $x \in P_h$,

with $0 \leq N \leq \mathcal{N}$ then we have the norm estimate

$$\begin{aligned}
C \left| \varphi(x_c) \int_{\mathfrak{P}} e^{\rho \cdot (x - x_c)} P_N(x - x_c) dx \right| &\leq |\Re \rho|^{-N-n} e^{-\delta_0 |\Re \rho| h/2} \\
&+ |\Re \rho|^{-N-n-\min(1, \alpha)} + |\Re \rho|^{-N-n/p'} \|\psi\|_{L^p(P_h)} \\
&+ |\rho| (1 + \|\psi\|_{H^2(B_{2R})}) \sup_{\partial \Omega \cap B(x_c, h)} \{|u - u^i|, |\nabla u - \nabla u^i|\} \\
&+ h^{-1} e^{-\delta_0 |\Re \rho| h} |\rho| (1 + \|\psi\|_{H^2(B_{2R})}) \|u - u^i\|_{H^2(B_{2R})} \quad (6.3)
\end{aligned}$$

where $1/p + 1/p' = 1$ and $C > 0$ depends on all the a-priori parameters $\mathcal{V}, k, \mathcal{P}, \mathcal{M}, \mathcal{N}, \mathcal{R}, \mathcal{F}, \alpha, \delta_0, n, p$.

Proof. The integral identity is a direct calculation using Proposition 6.1 with Q_h and then noting that $V = 0$ on $Q_h \setminus P_h$. For the others we will use the incomplete gamma functions

$$\gamma(s, x) = \int_0^x e^{-t} t^{s-1} dt, \quad \Gamma(s, x) = \int_x^\infty e^{-t} t^{s-1} dt$$

which satisfy $\gamma(s, x) \leq \Gamma(s) \leq [s-1]!$ and $\Gamma(s, x) \leq 2^s \Gamma(s) e^{-x/2}$. By a radial change of coordinates the first integral on the right has the upper bound

$$\begin{aligned}
\left| \int_{\mathfrak{P} \setminus P_h} e^{\rho \cdot (x - x_c)} P_N(x - x_c) dx \right| &\leq \int_{\mathfrak{P} \setminus P_h} e^{-\delta_0 |\Re \rho| |x - x_c|} \mathcal{P} |x - x_c|^N dx \\
&\leq \mathcal{P} \sigma(\mathbb{S}^{n-1}) \int_h^\infty e^{-\delta_0 |\Re \rho| r} r^{N+n-1} dr \\
&\leq \left(\frac{2}{\delta_0} \right)^{N+n} (N+n)! \mathcal{P} \sigma(\mathbb{S}^{n-1}) |\Re \rho|^{-N-n} e^{-\delta_0 |\Re \rho| h/2} \\
&\leq C_{\delta_0, \mathcal{N}, n, \mathcal{P}} |\Re \rho|^{-N-n} e^{-\delta_0 |\Re \rho| h/2}
\end{aligned}$$

for the first integral on the right.

For the next three integrals we use the following estimate. Let f, g be functions such that $|f(x)| \leq A |x - x_c|^B$ with $A \leq \mathcal{A}$, $B \leq \mathcal{B}$, and $g \in L^q$. Then

$$\begin{aligned}
\left| \int_{P_h} e^{\rho \cdot (x - x_c)} f(x) g(x) dx \right| &\leq A \left(\frac{\sigma(\mathbb{S}^{n-1}) \gamma(Bq' + n, \delta_0 p' |\Re \rho| h)}{(\delta_0 q' |\Re \rho|)^{Bq' + n}} \right)^{1/q'} \|g\|_{L^q(P_h)} \\
&\leq A \left(\frac{\sigma(\mathbb{S}^{n-1}) [Bq' + n]!}{(\delta_0 q' |\Re \rho|)^{Bq' + n}} \right)^{1/q'} \|g\|_{L^q(P_h)} \\
&\leq C_{\mathcal{A}, \mathcal{B}, n, \delta_0, q} |\Re \rho|^{-B-n/q'} \|g\|_{L^q(P_h)}
\end{aligned}$$

where $1/q + 1/q' = 1$. Choosing

- $q = \infty$, $\mathcal{A} = \mathcal{PM}$, $B = N + \alpha \leq \mathcal{N} + \alpha$,
- $q = \infty$, $\mathcal{A} = \mathcal{MR}$, $B = N + 1 \leq \mathcal{N} + 1$, and
- $q = p$, $\mathcal{A} = \mathcal{MF}$, $B = N \leq \mathcal{N}$

gives the three estimates

$$\begin{aligned} \left| \int_{P_h} e^{\rho \cdot (x - x_c)} \varphi_\alpha(x) P_N(x - x_c) dx \right| &\leq C_{\mathcal{P}, \mathcal{M}, \mathcal{N}, \alpha, n, \delta_0} |\Re \rho|^{-N-n-\alpha}, \\ \left| \int_{P_h} e^{\rho \cdot (x - x_c)} V(x) u_{N+1}^i(x) dx \right| &\leq C_{\mathcal{M}, \mathcal{R}, \mathcal{N}, n, \delta_0} |\Re \rho|^{-N-n-1}, \\ \left| \int_{P_h} e^{\rho \cdot (x - x_c)} V(x) u^i(x) \psi(x) dx \right| &\leq C_{\mathcal{M}, \mathcal{F}, \mathcal{N}, n, \delta_0, p} |\Re \rho|^{-N-n/p'} \|\psi\|_{L^p(P_h)}. \end{aligned}$$

Let us split the boundary integral into two pieces: $\partial Q_h = (\partial \Omega \cap B(x_c, h)) \cup (\Omega \cap S(x_c, h))$. For the first piece

$$\begin{aligned} \left| \int_{\partial \Omega \cap B(x_c, h)} (u_0 \partial_\nu(u - u^i) - (u - u^i) \partial_\nu u_0) d\sigma(x) \right| &\leq \sigma(\partial \Omega \cap B(x_c, h)) \cdot \\ &\cdot ((1 + |\rho|)(1 + \|\psi\|_{L^2(\partial \Omega \cap B(x_c, h))}) + \|\partial_\nu \psi\|_{L^2(\partial \Omega \cap B(x_c, h))}) \|u - u^i\|_{NF} \end{aligned}$$

where $\|f\|_{NF}$ denotes the maximum of $|f|$ and $|\nabla f|$ on $\partial \Omega \cap B(x_c, h)$. The estimate follows since $|\exp(\rho \cdot (x - x_c))| \leq 1$ in Ω . Next note that we have $\|\psi\|_{L^2(\partial \Omega \cap B(x_c, h))}, \|\partial_\nu \psi\|_{L^2(\partial \Omega \cap B(x_c, h))} \leq C_{\mathcal{V}, n} \|\psi\|_{H^2(B_{2R})}$. The upper bound for the trace-operator norm depends only on \mathcal{V} and n since $\partial \Omega \cap B(x_c, h)$ is a subset of a union of boundaries of sets of the form $B(x_c, 1) \cap H_j$ for some half-spaces $H_1, \dots, H_{\mathcal{V}}$. The trace norm is identical in each of these sets. Similarly we see that $\sigma(\partial \Omega \cap B(x_c, h)) \leq C_{\mathcal{V}, n}$ and the estimate follows then using $|\rho| \geq 1$.

Let us consider how the trace-norm depends on h in the mapping $H^1(B(x_c, h)) \rightarrow L^2(S(x_c, h))$. We do this by scaling the variables, for example by having $g(y) = f((y - x_c)/h + x_c)$. Now

$$\begin{aligned} \|f\|_{L^2(S(x_c, h))} &= h^{\frac{n-1}{2}} \|g\|_{L^2(S(x_c, 1))} \leq Ch^{\frac{n-1}{2}} \|g\|_{H^1(B(x_c, 1))} \\ &\leq Ch^{-\frac{1}{2}} \|f\|_{H^1(B(x_c, h))} \end{aligned}$$

since $\|g\|_{H^1(B(x_c, 1))} \leq h^{-n/2} \|f\|_{H^1(B(x_c, h))}$ because of $h \leq 1$. Then

$$\begin{aligned} \left| \int_{\Omega \cap S(x_c, h)} (u_0 \partial_\nu(u - u^i) - (u - u^i) \partial_\nu u_0) d\sigma(x) \right| &\leq C_n h^{-\frac{1}{2}} \|u - u^i\|_{H^2(B(x_c, h))} \cdot \\ &\cdot e^{-\delta_0 |\Re \rho| h} ((1 + |\rho|)(1 + \|\psi\|_{L^2(\Omega \cap S(x_c, h))}) + \|\partial_\nu \psi\|_{L^2(\Omega \cap S(x_c, h))}). \end{aligned}$$

Now, using again the trace-norm scaling as at the start of this paragraph, we see $\|\psi\|_{L^2(\Omega \cap S(x_c, h))}, \|\partial_\nu \psi\|_{L^2(\Omega \cap S(x_c, h))} \leq C_n h^{-1/2} \|\psi\|_{H^2(B_{2R})}$ and hence the final estimate follows by $|\rho| \geq 1$. \square

We will need a lower bound on the left-hand side of (6.3) to prove stability. This is nontrivial. In previous papers [3], [22] it is shown that the left-hand side does not vanish. We do need a quantitative version, for example of the form: given a polynomial P_N satisfying some a-priori conditions, the left hand side is greater than C which does not depend on P_N . This turns out to require a too fine analysis in the context of support probing. However we can avoid this because we assumed that $u'(x_c) \neq 0$, which implies that $P_N(x) \equiv u'(x_c)$ is constant.

Lemma 6.3. *Let $n \in \{2, 3\}$, $0 < 2\alpha_m < 2\alpha_M < 2\alpha' < \pi$ and $k > 0$. For $\mathcal{Q}, \mathfrak{P} \subset \mathbb{R}^n$ we say $(\mathcal{Q}, \mathfrak{P}) \in \mathcal{G}(\alpha_m, \alpha_M, \alpha', n)$ if the following are satisfied*

- (1) \mathcal{Q} is an open spherical cone,
- (2) \mathfrak{P} is an open convex polygonal cones,
- (3) \mathcal{Q} and \mathfrak{P} have a common vertex $x_c \in \mathbb{R}^n$,
- (4) $\mathfrak{P} \subset \mathcal{Q}$,
- (5) \mathcal{Q} has opening angle at most $2\alpha'$,
- (6) in 2D \mathfrak{P} has opening angle in $]2\alpha_m, 2\alpha_M[$,
- (7) in 3D \mathfrak{P} can be transformed to $]0, \infty[^3$ by a rigid motion.

If $(\mathcal{Q}, \mathfrak{P}) \in \mathcal{G}(\alpha_m, \alpha_M, \alpha', n)$, then there is $\tau_0 = kC(\alpha_m, \alpha_M, \alpha', n) > 0$, and $c = c(\alpha_m, \alpha_M, n) > 0$ with the following properties. There is a curve $\tau \mapsto \rho(\tau) \in \mathbb{C}^n$ (which depends on \mathcal{Q}) satisfying $\rho(\tau) \cdot \rho(\tau) + k^2 = 0$, $\tau = |\Re \rho(\tau)|$,

$$\Re \rho(\tau) \cdot (x - x_c) \leq -\cos \alpha' |\Re \rho(\tau)| |x - x_c|$$

for all $x \in \mathcal{Q}$ and such that if $\tau \geq \tau_0$ then

$$\left| \int_{\mathfrak{P}} e^{\rho(\tau) \cdot (x - x_c)} dx \right| \geq c\tau^{-n}.$$

Proof. We will prove the claim for $\zeta \cdot \zeta = 0$ instead of $\rho \cdot \rho + k^2 = 0$ first. Consider the cases $n = 2$ and $n = 3$ separately. Let $(\mathcal{Q}, \mathfrak{P}) \in \mathcal{G}(\alpha_m, \alpha_M, \alpha', 2)$. Then there is a rigid motion $M_{\mathfrak{P}}$ and $\alpha \in [2\alpha_m, 2\alpha_M]$ such that $M_{\mathfrak{P}}$ takes \mathfrak{P} to $\{x \in \mathbb{R}^2 \mid x_2 > 0, x_1 > ax_2\}$ where $a = 1/\tan \alpha$. We have $M_{\mathfrak{P}}x = R_{\mathfrak{P}}(x - x_c)$ for some rotation $R_{\mathfrak{P}}$. Denote $\xi = R_{\mathfrak{P}}\zeta$. Then

$$\int_{\mathfrak{P}} e^{\zeta \cdot (x - x_c)} dx = \int_0^\infty \int_{ax_2}^\infty e^{\xi \cdot y} dy_1 dy_2 = \frac{1}{\xi_1(\xi_2 + a\xi_1)}$$

if $\Re \xi_1 < 0$ and $\Re(\xi_2 + a\xi_1) < 0$. If $\zeta \cdot \zeta = 0$ and $|\Re \zeta| = 1$ then the same is true for ξ and so $|\xi| \leq \sqrt{2}$. Thus

$$\left| \int_{\mathfrak{P}} e^{\zeta \cdot (x - x_c)} dx \right| \geq \frac{1}{2(1 + |a|)} > 0$$

because $|a|$ can be estimated above by $1/\min |\tan \alpha|$, where the minimum is taken over $2\alpha_m \leq \alpha \leq 2\alpha_M$, and the limits are away from 0 and π .

The conditions $\Re \xi_1 < 0$ and $\Re(\xi_2 + a\xi_1) < 0$ are implied at once if

$$\Re \zeta \cdot (x - x_c) \leq -\cos \alpha' |x - x_c|$$

for all $x \in \mathcal{Q}$ as this means that the map $x \mapsto \exp(\Re \zeta \cdot (x - x_c))$ is exponentially decreasing in \mathcal{Q} , and a fortiori in \mathfrak{P} . We can now build ζ . Let $-\Re \zeta$ be the unit vector on the central axis of \mathcal{Q} to make the above inequality valid. Next choose $\Im \zeta$ such that $\Im \zeta \perp \Re \zeta$, $|\Im \zeta| = 1 = |\Re \zeta|$. This implies $\zeta \cdot \zeta = 0$.

Consider the 3D case now. Let $(\mathcal{Q}, \mathfrak{P}) \in \mathcal{G}(\alpha_m, \alpha_M, \alpha', 3)$. Then there is a rigid motion $M_{\mathfrak{P}}$ bringing \mathfrak{P} to $]0, \infty[^3$. We have $M_{\mathfrak{P}}x = R_{\mathfrak{P}}(x - x_c)$ for some rotation $R_{\mathfrak{P}}$. Denote again $\xi = R_{\mathfrak{P}}\zeta$. Then

$$\int_{\mathfrak{P}} e^{\zeta \cdot (x - x_c)} dx = \int_{]0, \infty[^3} e^{\xi \cdot y} dy = \frac{-1}{\xi_1 \xi_2 \xi_3}$$

as long as $\xi_j < 0$ for all j . As before, $\zeta \cdot \zeta = 0$ and $|\Re \zeta| = 1$ imply $|\xi| \leq \sqrt{2}$ and the lower bound of $2^{-3/2}$ for the integral. The conditions $\xi_j < 0$ follow from

$$\Re \zeta \cdot (x - x_c) \leq -\cos \alpha' |x - x_c|$$

in \mathcal{Q} . The choice of ζ is made as in the 2D case.

To recap, in both 2D and 3D, for any $(\mathcal{Q}, \mathfrak{P}) \in \mathcal{G}(\alpha_m, \alpha_M, \alpha', n)$ we found $\zeta \in \mathbb{C}^n$ satisfying $\zeta \cdot \zeta = 0$, $|\zeta| = 1$, $\Re \zeta \cdot (x - x_c) \leq -\cos \alpha' |x - x_c|$ for all $x \in \mathcal{Q}$ with x_c the vertex, and finally

$$\left| \int_{\mathfrak{P}} e^{\zeta \cdot (x - x_c)} dx \right| \geq 2C_{\alpha_m, \alpha_M, n} > 0.$$

Let us build the curve $\rho(\tau)$ next. Set

$$\rho(\tau) = \tau \Re \zeta + i \sqrt{\tau^2 + k^2} \Im \zeta.$$

It is easy to see that $\rho(\tau)/\tau \rightarrow \zeta$ as $\tau \rightarrow \infty$, and even easier to see that $\Re \rho(\tau) \cdot (x - x_c) \leq -\cos \alpha' |\Re \rho(\tau)| |x - x_c|$ for $x \in \mathcal{Q}$. Write $\mathcal{L}(\zeta) = \int_{\mathfrak{P}} \exp(\zeta \cdot (x - x_c)) dx$ to conserve space. Next we will quantify how far $\mathcal{L}(\rho(\tau)/\tau)$ is from $\mathcal{L}(\zeta)$. Ideally we want an estimate that does not depend on \mathcal{Q} or \mathfrak{P} .

If we set $f(r) = \exp((\Re \zeta + ir \Im \zeta) \cdot (x - x_c))$ then $f(1) = \exp(\zeta \cdot (x - x_c))$ and $f(\sqrt{1 + k^2/\tau^2}) = \exp(\rho(\tau)/\tau \cdot (x - x_c))$. By the mean value theorem

$$\left| f(1) - f\left(\sqrt{1 + \frac{k^2}{\tau^2}}\right) \right| \leq \sup_{1 < r < \sqrt{1 + \frac{k^2}{\tau^2}}} |f'(r)| \left| \sqrt{1 + \frac{k^2}{\tau^2}} - 1 \right|.$$

Note that $\sqrt{1 + k^2/\tau^2} - 1 \leq k/\tau$. Also $f'(r) = i \Im \zeta \cdot (x - x_c) f(r)$ and thus since $|\Im \zeta| = |\Re \zeta| = 1$ we have $|f'(r)| \leq |x - x_c| \exp(-\cos \alpha' |x - x_c|)$. Hence

$$\left| f(1) - f\left(\sqrt{1 + \frac{k^2}{\tau^2}}\right) \right| \leq \frac{k}{\tau} |x - x_c| e^{-\cos \alpha' |x - x_c|}.$$

We see finally that

$$\begin{aligned} \left| \mathcal{L}(\zeta) - \mathcal{L}\left(\frac{\rho(\tau)}{\tau}\right) \right| &= \left| \int_{\mathfrak{P}} (f(1) - f(\sqrt{1 + k^2/\tau^2})) dx \right| \\ &\leq \frac{k}{\tau} \int_{\mathfrak{P}} e^{-\cos \alpha' |x - x_c|} |x - x_c| dx \\ &\leq \sigma(\mathfrak{P} \cap \mathbb{S}^{n-1}) \frac{k}{\tau} \int_0^\infty e^{-\cos \alpha' r} r^{1+n-1} dr \leq C_{\alpha', n} k \tau^{-1} \end{aligned}$$

because we can estimate $\sigma(\mathfrak{P} \cap \mathbb{S}^{n-1}) \leq \sigma(\mathbb{S}^{n-1})$, and $\cos \alpha' > 0$ since $\alpha' < \pi/2$.

Now, it is easily seen that $\mathcal{L}(\rho(\tau)/\tau) = \tau^n \mathcal{L}(\rho(\tau))$. Recall that our choice of ζ implies that $|\mathcal{L}(\zeta)| \geq 2C_{\alpha_m, \alpha_M, n}$. By the triangle inequality

$$\begin{aligned} |\tau^n \mathcal{L}(\rho(\tau))| &= \left| \mathcal{L}\left(\frac{\rho(\tau)}{\tau}\right) \right| \geq |\mathcal{L}(\zeta)| - \left| \mathcal{L}(\zeta) - \mathcal{L}\left(\frac{\rho(\tau)}{\tau}\right) \right| \\ &> 2C_{\alpha_m, \alpha_M, n} - C_{\alpha', n} k \tau^{-1} \geq C_{\alpha_m, \alpha_M, n} > 0 \end{aligned}$$

if $\tau \geq C_{\alpha', n} k / C_{\alpha_m, \alpha_M, n}$ which is finite and depends only on the a-priori parameters. \square

7. COMPLEX GEOMETRICAL OPTICS SOLUTION

The construction of the CGO solutions in corner scattering was first shown in [3] and [22]. We continue the analysis involved by keeping track of what parameters the coefficients depend on. This involves us defining a “norm” for polygonal regions. We will first solve the Faddeev equation, then prove estimates for potentials supported on polytopes and finally build the complex geometrical optics solutions.

Lemma 7.1. *Let $s \geq 0$ and $1 \leq r \leq 2$ such that $1/r + 1/r' = 1$ and*

$$\frac{2}{n+1} \leq \frac{1}{r} - \frac{1}{r'} < \frac{2}{n}.$$

Let q be a measurable function such that the pointwise multiplier operator m_q maps $H_{r'}^s(\mathbb{R}^n) \rightarrow H_r^s(\mathbb{R}^n)$, and let $f \in H_r^s(\mathbb{R}^n)$.

Let $I_0 = (2M \|m_q\|_{H_{r'}^s \rightarrow H_r^s})^{2+n/r'-n/r}$, where $M = M(r, s, n) \geq 1$ is fixed in the proof. Then if $\rho \in \mathbb{C}^n$, $|\Im \rho| \geq I_0$ there is $\psi \in H_{r'}^s(\mathbb{R}^n)$ satisfying

$$(\Delta + 2\rho \cdot \nabla + q)\psi = f,$$

$$\|\psi\|_{H_{r'}^s(\mathbb{R}^n)} \leq 2M |\Im \rho|^{-(2+\frac{n}{r'}-\frac{n}{r})} \|f\|_{H_r^s(\mathbb{R}^n)}.$$

There is also $p \geq 2$ and a Sobolev embedding constant $E = E(s, n, r) \geq 1$ such that

$$\|\psi\|_{L^p(\mathbb{R}^n)} \leq EM |\Im \rho|^{-(2+\frac{n}{r'}-\frac{n}{r})} \|f\|_{H_r^s(\mathbb{R}^n)}.$$

We have the following observations about the choice of p and the decay rate of ψ compared to $|\Im \rho|^{-n/p}$.

- (1) *If $s > \frac{n}{r'}$ then $p = \infty$ and $2 + \frac{n}{r'} - \frac{n}{r} > \frac{n}{p}$,*
- (2) *if $s = \frac{n}{r'}$ then we may choose any finite p such that $\frac{1}{p} < \frac{2}{n} + \frac{1}{r'} - \frac{1}{r}$ which is positive, and then $2 + \frac{n}{r'} - \frac{n}{r} > \frac{n}{p}$,*
- (3) *if $\frac{n}{r} - 2 < s < \frac{n}{r'}$ then $s - \frac{n}{r'} = -\frac{n}{p}$ and $2 + \frac{n}{r'} - \frac{n}{r} > \frac{n}{p}$, and finally*
- (4) *if $s \leq \frac{n}{r} - 2$ then $s - \frac{n}{r'} = -\frac{n}{p}$ but $2 + \frac{n}{r'} - \frac{n}{r} \leq \frac{n}{p}$.*

Lastly, if $f \in L_{loc}^2$ and $q \in L_{loc}^\infty$ then given any bounded domain, for example B_{3R} , we have the elliptic regularity estimate

$$\|\psi\|_{H^2(B_{2R})} \leq C_R (\|f\|_{L^2(B_{3R})} + (1 + |\rho|^2 + \|q\|_{L^\infty(B_{3R})}) \|\psi\|_{L^p(\mathbb{R}^n)})$$

where C_R depends only on R .

Proof. Fix $M < \infty$ as the ρ -independent constant in the estimate

$$\|f\|_{L^{r'}(\mathbb{R}^n)} \leq M |\Im \rho|^{n(1/r-1/r')-2} \|(\Delta + 2\rho \cdot \nabla)f\|_{L^r(\mathbb{R}^n)}$$

by [14] or in Theorem 5.4 in the notes [26]. By Proposition 3.3 in [22] the equation

$$(\Delta + 2\rho \cdot \nabla + q)\psi = f$$

has a solution $\psi \in H_{r'}^s(\mathbb{R}^n)$ when $|\Im \rho| \geq I_0$. Moreover it satisfies

$$\|\psi\|_{H_{r'}^s(\mathbb{R}^n)} \leq 2M |\Im \rho|^{-(2+n/r'-n/r)} \|f\|_{H_r^s(\mathbb{R}^n)}.$$

Sobolev embedding implies the L^p estimates in the four cases of the statement. Note that in each case we have $p \geq r' \geq 2$.

The elliptic regularity estimate needs some work. First assume that $G \in H^s(\mathbb{R}^n)$, $F \in H^s(\mathbb{R}^n)$ and $(\Delta + 2\rho \cdot \nabla)G = F$. Then

$$\begin{aligned} \|G\|_{H^{s+2}(\mathbb{R}^n)} &= \left\| (1 + |\xi|^2)^{s/2} (1 + |\xi|^2) \hat{G} \right\|_{L^2(\mathbb{R}^n)} \\ &= \left\| (1 + |\xi|^2)^{s/2} (\hat{G} + 2i\rho \cdot \xi \hat{G} - \hat{F}) \right\|_{L^2(\mathbb{R}^n)} \\ &\leq \|G\|_{H^s(\mathbb{R}^n)} + \|F\|_{H^s(\mathbb{R}^n)} + 2 \left\| (1 + |\xi|^2)^{s/2} \rho \cdot \xi \hat{G} \right\|_{L^2(\mathbb{R}^n)} \end{aligned}$$

because $(-|\xi|^2 + 2i\rho \cdot \xi)\hat{G} = \hat{F}$. By looking at what happens when $|\xi|$ is larger or smaller than $3|\rho|$ we see that $|\rho \cdot \xi| \leq -|\xi|^2 + 2i\rho \cdot \xi + 3|\rho|^2$. Hence

$$\|G\|_{H^{s+2}(\mathbb{R}^n)} \leq 3\|F\|_{H^s(\mathbb{R}^n)} + (1 + 6|\rho|^2) \|G\|_{H^s(\mathbb{R}^n)}. \quad (7.1)$$

Now let $\chi, \tilde{\chi} \in C_0^\infty(\mathbb{R}^n)$ such that $\tilde{\chi} \equiv 1$ on $\text{supp } \chi$. Assume that $f \in L_{loc}^2$ and $q \in L_{loc}^\infty$. Next

$$(\Delta + 2\rho \cdot \nabla)(\chi\psi) = \chi(f - q\psi) + 2\nabla\chi \cdot \nabla(\tilde{\chi}\psi) + (\Delta\chi + 2\rho \cdot \nabla\chi)\psi \quad (7.2)$$

in the distribution sense. We have $q\psi \in L_{loc}^p$, $p \geq 2$ so $\chi q\psi \in L^2(\mathbb{R}^n)$. Similarly $\tilde{\chi}\psi \in L^2(\mathbb{R}^n)$ and so $\nabla\chi \cdot \nabla(\tilde{\chi}\psi) \in H^{-1}(\mathbb{R}^n)$. The last term on the right-hand side is in $L^2(\mathbb{R}^n)$. By absorbing all the norms of $\chi, \tilde{\chi}$ into a constant we get the estimate

$$C_{\chi, \tilde{\chi}, p} (\|f\|_{L^2(\text{supp } \chi)} + (1 + |\rho| + \|q\|_{L^\infty(\text{supp } \chi)}) \|\psi\|_{L^p(\mathbb{R}^n)})$$

for the $H^{-1}(\mathbb{R}^n)$ -norm of the right-hand side. By (7.1) and since $\psi \in L^p$, $p \geq 2$,

$$\|\chi\psi\|_{H^1(\mathbb{R}^n)} \leq \tilde{C}_{\chi, \tilde{\chi}, p} (\|f\|_{L^2(\text{supp } \chi)} + (1 + |\rho| + |\rho|^2 + \|q\|_{L^\infty(\text{supp } \chi)}) \|\psi\|_{L^p(\mathbb{R}^n)})$$

and this is true no matter the choice of $\chi, \tilde{\chi} \in C_0^\infty(\mathbb{R}^n)$, $\tilde{\chi} \equiv 1$ on $\text{supp } \chi$.

Consider the bounded domain B_{2R} now. Take a chain of cut-off functions $\chi, \tilde{\chi}, \bar{\chi} \in C_0^\infty(B_{3R})$ such that $\bar{\chi} \equiv 1$ on $\text{supp } \tilde{\chi}$, $\tilde{\chi} \equiv 1$ on $\text{supp } \chi$ and finally $\chi \equiv 1$ on B_{2R} . Then $\chi\psi \in H^2(\mathbb{R}^n)$ according to (7.1) if the right-hand side of (7.2) is in $L^2(\mathbb{R}^n)$. But this is indeed true by going through the previous paragraph while substituting $(\tilde{\chi}, \bar{\chi})$ for $(\chi, \tilde{\chi})$. This gives the final estimate

$$\begin{aligned} \|\psi\|_{H^2(B_{2R})} &\leq \|\chi\psi\|_{H^2(\mathbb{R}^n)} \\ &\leq \mathcal{C}_{\chi, \tilde{\chi}, \bar{\chi}, p} (\|f\|_{L^2(\text{supp } \tilde{\chi})} + (1 + |\rho| + |\rho|^2 + \|q\|_{L^\infty(\text{supp } \tilde{\chi})}) \|\psi\|_{L^p(\mathbb{R}^n)}) \end{aligned}$$

which can be bounded above by the estimate of the statement. Note that the test functions can be chosen based exclusively on the set B_{2R} , and their norms have a finite supremum while p explores the whole set $[2, \infty]$. Hence the constant can be made to depend only on R . \square

We will next prove norm estimates for a potential consisting of a Hölder-continuous function multiplying the characteristic function of a polytope. For a clearer notation we define a multiplier norm for a polytope first.

Definition 7.2. A set $P \subset \mathbb{R}^n$ is a *bounded open polytope* if P is bounded, open and \bar{P} is a finite union of finite intersections of closed half-spaces.

Definition 7.3. Let $P \subset \mathbb{R}^n$ be an bounded open polytope. We say a collection $\{H_{jl} \mid j = 1, \dots, J, l = 1, \dots, L_j\}$ of half-spaces is a *triangulation* of P if $J \in \mathbb{N}$, $L_1, \dots, L_J \in \mathbb{N}$, $H \subset H_{jl} \subset \overline{H}$ for some open half-space $H \in \mathbb{R}^n$, the intersections $\bigcap_l H_{jl}$ are disjoint for different j , and

$$P = \bigcup_{j=1}^J \bigcap_{l=1}^{L_j} H_{jl}.$$

If $s \in \mathbb{R}$ and $1 \leq r < \infty$ let $C_{s,r} \in \mathbb{R} \cup \{+\infty\}$ be the norm of the map $H_r^s(\mathbb{R}^n) \rightarrow H_r^s(\mathbb{R}^n)$, $f \mapsto \chi_H f$, where $H \subset \mathbb{R}^n$ is a half-space. Then by $\|P\|_{T(s,r)}$ we mean

$$\|P\|_{T(s,r)} = \inf \left\{ \sum_{j=1}^J C_{s,r}^{L_j} \mid (H_{jl})_{j,l} \text{ is a triangulation of } P \right\}. \quad (7.3)$$

Lemma 7.4. Let $P \subset \mathbb{R}^n$ be a bounded open polytope, $s \geq 0$, $r \geq 1$ and $sr < 1$. Then $\|P\|_{T(s,r)} < \infty$ and $\|\chi_P f\|_{H_r^s(\mathbb{R}^n)} \leq \|P\|_{T(s,r)} \|f\|_{H_r^s(\mathbb{R}^n)}$. Moreover we have $\|P\|_{T(s_0,r)} \leq \|P\|_{T(s_1,r)}$ if $s_0 \leq s_1$.

Proof. By definition P has a finite triangulation of let us say $m < \infty$ simplices. Each simplex in \mathbb{R}^n is the intersection of $n+1$ half-spaces. By Triebel [28], Section 2.8.7, the map $f \mapsto \chi_H f$ is bounded in $H_r^s(\mathbb{R}^n)$ under the conditions for s and r given. Hence $\|P\|_{T(s,r)} \leq m C_{s,r}^{n+1} < \infty$. If $(H_{jl})_{j,l}$ is a triangulation, then the intersections $\bigcap_{l=1}^{L_j} H_{jl}$ are disjoint, so $\chi_P = \sum_{j=1}^J \prod_{l=1}^{L_j} \chi_{H_{jl}}$ and thus $\|\chi_P f\|_{H_r^s(\mathbb{R}^n)} \leq \sum_{j=1}^J C_{s,r}^{L_j} \|f\|_{H_r^s(\mathbb{R}^n)}$. The multiplier estimate follows by taking the infimum over all triangulations. The last claim follows since complex interpolation of Sobolev spaces implies that $C_{s_0,r} \leq C_{s_1,r}$ if $s_0 \leq s_1$. \square

Lemma 7.5. Let $V = \chi_P \varphi$ with $P \subset B_R$ an open polytope and $\varphi \in C^\alpha(\mathbb{R}^n)$ with $\alpha > 0$. Let $0 \leq s < \alpha$, $1 \leq r \leq 2$ and $sr < 1$. Then $V \in H_r^s(\mathbb{R}^n)$,

$$\|V\|_{H_r^s(\mathbb{R}^n)} \leq C_{\alpha,s,r,R} \|P\|_{T(s,r)} \|\varphi\|_{C^\alpha(\mathbb{R}^n)}$$

and

$$\|V f\|_{H_r^s(\mathbb{R}^n)} \leq C_{\alpha,s,r,R} \|P\|_{T(s,r)} \|\varphi\|_{C^\alpha(\mathbb{R}^n)} \|f\|_{H_{r'}^s(\mathbb{R}^n)}$$

where $1/r + 1/r' = 1$ and $\|P\|_{T(s,r)}$ is defined in Definition 7.3.

Proof. Let $\Phi \in C_0^\infty$ be such that $\Phi = 1$ on B_R . Then we have the representation

$$V = \chi_P \varphi \Phi$$

which will help us prove the estimates.

By the last corollary of Section 4.2.2 in [29] there is a finite upper bound $C_{\alpha,s,r}$ for the pointwise multiplier operator norm of any C^α function multiplying in $H_r^s(\mathbb{R}^n)$ when $s < \alpha$. Then the first claim

$$\|V\|_{H_r^s(\mathbb{R}^n)} \leq C_{\alpha,s,r} \|\varphi\|_{C^\alpha(\mathbb{R}^n)} \|\chi_P \Phi\|_{H_r^s(\mathbb{R}^n)} \leq C_{\alpha,s,r,\Phi} \|P\|_{T(s,r)} \|\varphi\|_{C^\alpha(\mathbb{R}^n)}$$

follows from Lemma 7.4 since $\|P\|_{T(s,r)} < \infty$ by $s \geq 0$, $r \geq 1$ and $sr < 1$.

By [22] Proposition 3.5 or [2] Theorem 7.5 the product of a $H_{r'}^s(B_R)$ and $H_{r/(2-r)}^s(B_R)$ function is in $H_r^s(B_R)$ when $s \geq 0$ and $1 \leq r \leq 2$. According

to [29], we know that C^α -functions are pointwise multipliers for $H_{r/(2-r)}^s$ too. The last claim

$$\begin{aligned} \|Vf\|_{H_r^s(\mathbb{R}^n)} &\leq \|P\|_{T(s,r)} \|\varphi\Phi f\|_{H_r^s(B_R)} \\ &\leq M_{s,r} \|P\|_{T(s,r)} \|\varphi\Phi\|_{H_{\frac{r}{2-r}}^s(B_R)} \|f\|_{H_{r'}^s(B_R)} \\ &\leq C_{\alpha,s,r,\Phi} \|P\|_{T(s,r)} \|\varphi\|_{C^\alpha(\mathbb{R}^n)} \|f\|_{H_{r'}^s(\mathbb{R}^n)} \end{aligned}$$

follows then because V is supported in $\overline{B_R}$. \square

We are now ready to specialize previous lemmas into proving the existence of the complex geometrical optics solutions in the context of corner scattering in two and three dimensions.

The conditions on α follow from the various requirements: For the half-space multipliers we needed $sr < 1$ and $s < \alpha$. To have good enough error decay estimates for ψ from Lemma 7.1 we need $s > n/r - 2$. Combining these gives $n - 2r < sr < 1$ i.e. $r > (n - 1)/2$. On the other hand we must have $1/r - 1/r' \geq 2/(n + 1)$ i.e. $r \leq 2(n + 1)/(n + 3)$ in Lemma 7.1. These two inequalities have solutions only when $n \in \{2, 3\}$.

Moreover we have less requirements on α by making s and thus $n/r - 2$ as small as possible. This happens when r is largest, $r = 2(n + 1)/(n + 3)$.

Proposition 7.6. *Let $n \in \{2, 3\}$ and $0 \leq s < 5/6$ in 2D or $1/4 < s < 3/4$ in 3D. Let $\varphi \in C^\alpha(\mathbb{R}^n)$ with $\alpha > s$ and $\|\varphi\|_{C^\alpha} \leq \mathcal{M}$. Let $P \subset B_R$ be an open polytope, $r = 2(n + 1)/(n + 3)$, and assume that $\|P\|_{T(s,r)} \leq \mathcal{D}$.*

Let $k > 0$ and set $V = \chi_P \varphi$. Then there is $p \geq 2$ and $C_{\alpha,s,n,R} < \infty$ with the following properties. If $\rho \in \mathbb{C}^n$, $\rho \cdot \rho + k^2 = 0$, $|\Im \rho| \geq (C_{\alpha,s,n,R} k^2 \mathcal{D} \mathcal{M})^{(n+1)/2}$, then there is $\psi \in L^p(\mathbb{R}^n)$ such that $u_0(x) = \exp(\rho \cdot x)(1 + \psi(x))$ satisfies

$$(\Delta + k^2(1 + V))u_0 = 0$$

in \mathbb{R}^n , and

$$\|\psi\|_{L^p(\mathbb{R}^n)} \leq C_{\alpha,s,n,R} k^2 \mathcal{D} \mathcal{M} |\Im \rho|^{-n/p-\beta}$$

with $\beta = \beta(s, n) > 0$. Moreover $\psi \in H^2(B_{2R})$ with norm estimate

$$\|\psi\|_{H^2(2R)} \leq C_{\alpha,s,n,R} (1 + |\rho|^2 + (1 + k^2)\mathcal{M}).$$

Proof. Set $q = k^2 V$ and $f = -k^2 V$. Now $0 \leq s < \alpha$, $1 \leq r \leq 2$ and $sr < 1$, so by Lemma 7.5 we have

$$\|f\|_{H_r^s(\mathbb{R}^n)}, \|m_q\|_{H_{r'}^s \rightarrow H_r^s} \leq C_{\alpha,s,n,R} k^2 \|P\|_{T(s,r)} \|\varphi\|_{C^\alpha(\mathbb{R}^n)}$$

where m_q is the pointwise multiplier operator.

We have $1/r - 1/r' = 2/(n + 1)$, $r \leq 2$. The lower bound for $|\Im \rho|$ matches Lemma 7.1 so we have existence of ψ . The condition $s > n/r - 2$ that's required for the good enough error term decay is also satisfied by our a-priori requirements on s .

For the H^2 -norm estimate note that $I_0 = (C_{\alpha,s,n,R} k^2 \mathcal{D} \mathcal{M})^{(n+1)/2}$ and the bound for $\|f\|_{H_r^s}$ imply that $\|\psi\|_p \leq C_{s,n}$. We also see that $\|f\|_{L^2} \leq C_{n,R} \mathcal{M}$ by its definition. \square

8. STABILITY PROOFS

The proofs of the following two lemmas are in the appendix.

Lemma 8.1. *Let $P, P' \subset \mathbb{R}^2$ be two open bounded convex polygons. Let Q be the convex hull of $P \cup P'$. If x_c is a vertex of P such that $d(x_c, P') = d_H(P, P')$, where d_H gives the Hausdorff distance,*

$$d_H(P, P') = \max \left(\sup_{x \in P} d(x, P'), \sup_{x' \in P'} d(P, x') \right),$$

then x_c is a vertex of Q . If the angle of P at x_c is α , then the angle of Q at x_c is at most $(\alpha + \pi)/2 < \pi$.

Lemma 8.2. *Let $P, P' \subset \mathbb{R}^3$ be two open cuboids. Let Q be the convex hull of $P \cup P'$. If x_c is a vertex of P such that $d(x_c, P') = d_H(P, P')$, where d_H gives the Hausdorff distance,*

$$d_H(P, P') = \max \left(\sup_{x \in P} d(x, P'), \sup_{x' \in P'} d(P, x') \right),$$

then x_c is a vertex of Q . The latter can also fit inside an open spherical cone \mathcal{Q} with vertex x_c and opening angle $2\alpha' < \pi$. Here α' is independent of P and P' or their location.

We are ready to proof the final theorem whose statement is on page 5.

Proof of Theorem 3.1. By lemmas 8.1 and 8.2 and possibly switching the symbols P and P' (and their associated waves and potentials) we may assume that $\mathfrak{h} = d(x_c, P')$ with x_c a vertex of P . We will use the total wave u' of the second potential V' as a “local incident wave” in the neighbourhood of x_c . This is allowed since $(\Delta + k^2)u' = 0$ there since $V' = 0$ around x_c .

The potentials V and V' give well-posed scattering. Denote the L^2 -norm of the difference of the far-field patterns by $\varepsilon = \|u_\infty^s - u_\infty'^s\|_{L^2(\mathbb{S}^{n-1})}$. Use Proposition 5.10. If Q is the convex hull of $P \cup P'$ then

$$\sup_{\partial Q} |u^s - u'^s| + |\nabla(u^s - u'^s)| \leq \frac{C}{\sqrt{\ln \ln \frac{S}{\varepsilon}}}$$

when $\varepsilon < \varepsilon_m$. Here C and ε_m depend only on the a-priori parameters. Denote the right-hand side by $\delta(\varepsilon)$ to conserve space in formulas.

Let \mathfrak{Q} be the polygonal cone generated by Q at x_c . By lemmas 8.1 and 8.2 there is an open spherical cone $\mathcal{Q} \supset \mathfrak{Q} \supset Q$ with vertex x_c having opening angle at most $2\alpha' = 2\alpha'(\alpha_m, \alpha_M) < \pi$. Let \mathfrak{P} be the cone generated by P at its vertex x_c . Remember for later that $(\mathcal{Q}, \mathfrak{P}) \in \mathcal{G}(\alpha_m, \alpha_M, \alpha', n)$ using the notation from Lemma 6.3.

Let $h = \min(\ell, \mathfrak{h})$ and it is enough to consider the case $h > 0$. We have $P \cap B(x_c, h) = \mathfrak{P} \cap B(x_c, h)$ and $Q \cap B(x_c, h) = \mathfrak{Q} \cap B(x_c, h)$. Denote the former by P_h and the latter by Q_h . We also have $P_h \cap P' = Q_h \cap P' = \emptyset$. We want to use Corollary 6.2 with u' instead of u^i next.

The conditions from type 2 admissibility imply that we have $N = 0$, $P_N(x) \equiv u'(x_c) \neq 0$ and the other conditions of that corollary. Hence there is a constant C depending only on a-priori parameters such that if $1/p + 1/p' =$

1, then

$$\begin{aligned}
C \left| \varphi(x_c) \int_{\mathfrak{P}} e^{\rho \cdot (x-x_c)} u'(x_c) dx \right| &\leq |\Re \rho|^{-n} e^{-\delta_0 |\Re \rho| h/2} \\
&+ |\Re \rho|^{-n-\min(1,\alpha)} + |\Re \rho|^{-n/p'} \|\psi\|_{L^p(P_h)} \\
&+ |\rho| (1 + \|\psi\|_{H^2(B_{2R})}) \delta(\varepsilon) \\
&+ h^{-1} e^{-\delta_0 |\Re \rho| h} |\rho| (1 + \|\psi\|_{H^2(B_{2R})})
\end{aligned} \tag{8.1}$$

whenever $u_0 \in H^2(Q_h)$ satisfies $(\Delta + k^2(1+V))u_0 = 0$,

$$u_0(x) = e^{\rho \cdot (x-x_c)} (1 + \psi(x)),$$

$\psi \in L^p(Q_h)$ with $\rho \in \mathbb{C}^n$, $|\rho| \geq 1$ and $\Re \rho \cdot (x - x_c) \leq -\delta_0 |\Re \rho| |x - x_c|$ for some $\delta_0 > 0$ and any $x \in Q_h$.

Recall that $(\mathcal{Q}, \mathfrak{P}) \in \mathcal{G}(\alpha_m, \alpha_M, \alpha', n)$, and hence we may use Lemma 6.3. It gives us constants $\tau_0 = \tau_0(k, \alpha_m, \alpha_M, \alpha', n)$, $c = c(\alpha_m, \alpha_M, n) > 0$ and a curve $\tau \mapsto \rho(\tau) \in \mathbb{C}^n$ satisfying the condition required on ρ above with $\delta_0 = \cos \alpha' > 0$, $\tau = |\Re \rho(\tau)|$, $\rho(\tau) \cdot \rho(\tau) + k^2 = 0$ and

$$\left| \int_{\mathfrak{P}} e^{\rho \cdot (x-x_c)} u'(x_c) dx \right| \geq c |u'(x_c)| \tau^{-n} \tag{8.2}$$

whenever $\tau \geq \tau_0$.

If $\tau \geq \max(\tau_0, C_0)$, with the constant C_0 depending on a-priori parameters and arising from Proposition 7.6, then the latter gives existence of u_0 and ψ required above. Moreover it gives the estimates

$$\|\psi\|_{L^p(\mathbb{R}^n)} \leq C |\Im \rho|^{-n/p-\beta}$$

for some $\beta = \beta(s, n) > 0$ and

$$\|\psi\|_{H^2(B_{2R})} \leq C(1 + |\rho|^2)$$

where C again depends only on the a-priori parameters.

We have all the fundamental estimates now. Let us apply them. We have $\exp(-x) \leq x^{-1}$ and $\exp(-x) \leq (n+4)! x^{-n-4}$ for all $x > 0$. Also, since $\rho(\tau) \cdot \rho(\tau) + k^2 = 0$, we get $|\rho(\tau)| = \sqrt{k^2 + 2\tau^2}$. By taking a new lower bound for τ , for example $\tau \geq k$, we may assume $|\rho(\tau)| \leq \sqrt{3}\tau$. Hence we can estimate

$$\begin{aligned}
|\Re \rho|^{-n} e^{-\delta_0 |\Re \rho| h/2} &\leq C |\Re \rho|^{-n-1} h^{-1}, \\
|\Re \rho|^{-n/p'} \|\psi\|_{L^p} &\leq C |\Re \rho|^{-n-\beta}, \\
|\rho| (1 + \|\psi\|_{H^2(B_{2R})}) \delta(\varepsilon) &\leq C |\Re \rho|^3 \delta(\varepsilon), \\
h^{-1} e^{-\delta_0 |\Re \rho| h} |\rho| (1 + \|\psi\|_{H^2(B_{2R})}) &\leq C |\Re \rho|^{-n-1} h^{-1}
\end{aligned}$$

in (8.1). Divide the new constants to the left hand side, take the lower bound (8.2) into account and use the a-priori assumption $|u'(x)| \geq c > 0$ in $B_R \setminus P'$. Finally, using $h \leq 1$, we get

$$ch |\varphi(x_c)| \leq \tau^{-\min(1,\alpha,\beta)} + \tau^3 h \delta(\varepsilon) \tag{8.3}$$

as long as $\tau \geq \max(\tau_0, C_0, k)$ and $h = \min(\ell, \mathfrak{h})$.

Setting $\tau = \tau_e$ with

$$\tau_e = \left(\frac{1}{h\delta(\varepsilon)} \right)^{\frac{1}{3+\min(1,\alpha,\beta)}}$$

makes both terms on the right hand side of (8.3) equal (which is the minimum modulo constants), and the inequality becomes

$$ch |\varphi(x_c)| \leq 2(h\delta(\varepsilon))^{\frac{\min(1,\alpha,\beta)}{3+\min(1,\alpha,\beta)}}. \quad (8.4)$$

Note that if ε is small enough, then

$$(\delta(\varepsilon))^{-\frac{1}{3+\min(1,\alpha,\beta)}} \geq \max(\tau_0, C_0, k),$$

and since $h \leq 1$, we have $\tau_e \geq \max(\tau_0, C_0, k)$ then. Solving for h in (8.4) gives

$$\min(\ell, \mathfrak{h}) = h \leq C \left(\frac{\delta(\varepsilon)^\gamma}{|\varphi(x_c)|} \right)^{1/(1-\gamma)}$$

for $\gamma = \min(1, \alpha, \beta)/(3 + \min(1, \alpha, \beta)) \in (0, 1)$.

By the type 1 admissibility of V we have $|\varphi(x_c)| \geq \mu > 0$. Hence if ε is again small enough (now also depending on μ), then the right-hand side is smaller than ℓ , and so $\min(\ell, \mathfrak{h}) = \mathfrak{h}$. Writing out the definition of $\delta(\varepsilon)$ gives

$$\mathfrak{h} \leq C \left(\ln \ln \frac{S}{\varepsilon} \right)^{-\eta}$$

where $\eta = \gamma/(2 - 2\gamma) > 0$ depends on α, β , and thus only on the a-priori parameters. \square

Proof of Theorem 3.2. The proof uses the same lemmas and propositions as the proof of Theorem 3.1. Now instead of having two non-trivial potentials V and V' , we choose the following: $P' = \emptyset$, $V' \equiv 0$. This implies that $u' = u^i$, $u'^s \equiv 0$, $u_\infty^s \equiv 0$ among others. In particular $V' \equiv 0$ is admissible of type 2.

Proceed as in the proof of Theorem 3.1, except that choose $h = \ell$ instead of $h = \min(\ell, d_H(P, P'))$. Up to showing (8.4) none of the constants depend on μ or ℓ . Using the notation from that proof, let $\varepsilon_{\min} \leq \varepsilon_m$, the latter being from Proposition 5.10, be such that

$$(\delta(\varepsilon_{\min}))^{-\frac{1}{3+\min(1,\alpha,\beta)}} \geq \max(\tau_0, C_0, k).$$

Since $\ell \leq 1$, we have $\tau_e \geq \max(\tau_0, C_0, k)$ if $\varepsilon \leq \varepsilon_{\min}$. Thus, in this case, solving for ε in (8.4) gives

$$\|u_\infty^s\|_{L^2(\mathbb{S}^{n-1})} = \varepsilon \geq \frac{S}{\exp \exp(C\ell^{-\gamma} |\varphi(x_c)|^{-\gamma-2})}$$

for $\gamma = 6/\min(1, \alpha, \beta) \geq 6$ and a constant C depending on a-priori data but not ℓ . If on the other hand $\varepsilon > \varepsilon_{\min}$ the claim is immediately true. \square

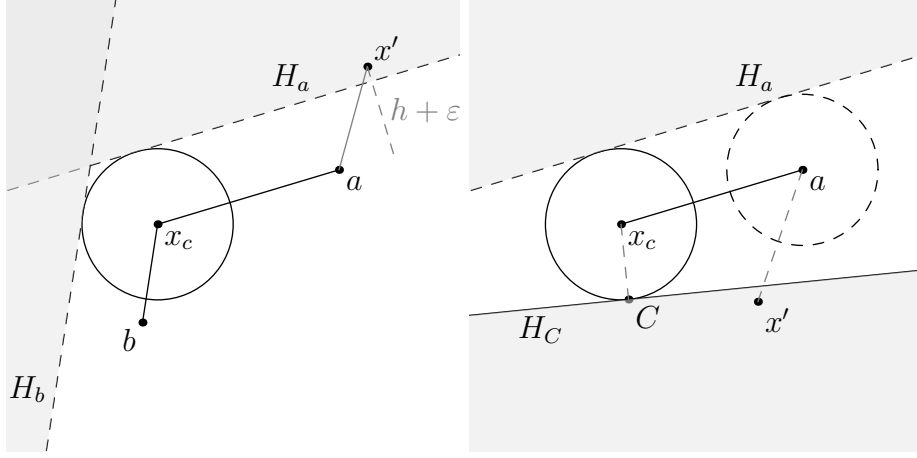


FIGURE 1. a) $P' \subset H_a^{\mathbb{C}} \cap H_b^{\mathbb{C}} \cap H_C$, b) ray x_c to a must meet H_C

9. APPENDIX

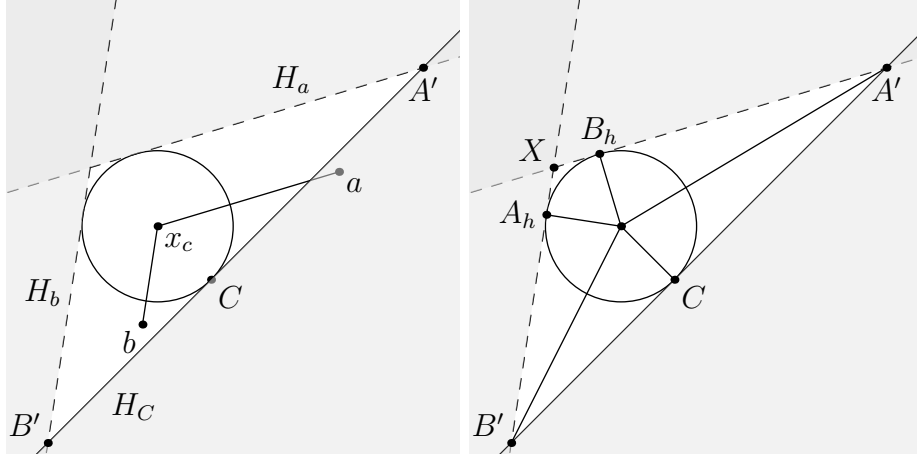
Proof of Lemma 8.1. Let a and b be the vertices of P on the adjacent edges to x_c . Let $C \in \overline{P'}$ be any point such that $d(x_c, C) = d_H(P, P')$, and let $h = d_H(P, P')$. Consider the circle $S(x_c, h)$. Let H_a be an open half-plane tangent to $S(x_c, h)$, parallel to the segment $x_c a$ and such that it is on the opposite side of $x_c a$ than b . Construct H_b similarly. See Figure 1a. Let H_C be the closed half-space tangent to $S(x_c, h)$ at C with $x_c \notin H_C$.

Let $x' \in P'$. If $x' \in H_a$, then $d(x', P) \geq d(x', \ell_{x_c, a}) > h$ where $\ell_{x_c, a}$ is a line through x_c and a . This follows from the convexity of P : the polygon is contained in the cone with vertex x_c and edges defined by a and b . Thus $d_H(P, P') \geq d(x', P) > h = d_H(P, P')$, a contradiction. Similarly for $x' \in H_b$. Consider H_C next: the convexity of P' implies that the segment $x' C$ belongs to $\overline{P'}$. If $x' \notin H_C$, then there is $y' \in x' C \cap B(x_c, h)$ by the non-tangency of $x' C$. Then $y' \in \overline{P'}$ and $d(x_c, y') < h$ so $d_H(P, P') < h$, a contradiction again. Thus we see that $P' \subset H_a^{\mathbb{C}} \cap H_b^{\mathbb{C}} \cap H_C$.

Next, H_C must be distance h from a : if it were not, then for any $x' \in P'$ we have $d(a, x') \geq d(a, H_C) > h$ since $P' \subset H_C$ as was shown above. Hence ∂H_C and ∂H_a are either parallel (a case we will skip in this proof) or meet at a point A' , in which case the ray from x_c towards a intersects H_C . Do the same for b to get B' . See Figure 1b. This means that $S(x_c, h)$ is the incircle of the triangle formed by H_a , H_b and H_C .

We can now see that x_c is a vertex of Q . First of all $x_c \in \overline{Q}$ since $x_c \in \overline{P}$. Also, P is inside the angle $ax_c b$ and P' inside the angle $A'x_c B'$, which is obviously less than π . Thus x_c is a vertex of Q . Moreover its angle is at most $\angle A'x_c B'$. See Figure 2a.

Let X be the intersection of ∂H_a and ∂H_b . This is a well-defined point since $0 < \angle ax_c b < \pi$. We have $\angle A'XB' = \angle ax_c b = \alpha$ by parallel transport of $x_c a$ to XA' and $x_c b$ to XB' . Let the perpendiculars from x_c to XA' , $A'B'$, $B'X$ have base points B_h , C , A_h , respectively. See Figure 2b. Then $\angle A_h x_c B_h = \pi - \alpha$, $\angle B_h x_c A' = \angle A'x_c C$ and $\angle Cx_c B' = \angle B'x_c A_h$. This

FIGURE 2. a) $S(x_c, h)$ is an incircle, b) solving $\angle A'x_cB'$

implies that $\angle A'x_cB' = (\alpha + \pi)/2$ at once since the sum of all of these angles is 2π . \square

Proof of Lemma 8.2. The proof proceeds as in the proof of Lemma 8.1. We can choose coordinates such that $x_c = \bar{0}$ and the three edges of P starting from x_c lie on the positive coordinate axes having unit vectors e_1 , e_2 and e_3 . Let $h = d(x_c, C) = d_H(P, P')$ for some $C \in P'$.

If we set $H_j = \{x \mid x \cdot e_j < -h\}$, then as in the 2D proof, we see that $P' \subset H_j^c$. Similarly, if H_C is the closed half-space tangent to $S(x_c, h)$ at C , we see that $P' \subset H_C$. Hence $P' \subset H_1^c \cap H_2^c \cap H_3^c \cap H_C$.

If $C_3 < 0$, i.e. it is on the lower hemisphere of $S(x_c, h)$, then there is $x \in P$ with $d(x, C) > h = d_H(P, P')$. Just take any x on the axis with $x_3 > 0$. The contradiction, seen also if $C_1 < 0$ or $C_2 < 0$, forces C to be on the closed spherical triangle $T = \{x \mid |x| = 1, x_j \geq 0\}$.

Now, no matter where $C \in T$ is, recalling that $P' \subset H_1^c \cap H_2^c \cap H_3^c \cap H_C$, it is easy to see that

$$\sup_{A, B \in P \cup P'} \angle Ax_cB < \pi$$

and hence that $H_1^c \cap H_2^c \cap H_3^c \cap H_C$ fits inside an spherical cone that does not contain a plane. Moreover the minimal required angle of the spherical cone depends continuously on the location of $C \in T$. Compactness of the latter implies the claim. \square

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